THE GENERAL SPACE-ENERGY NEUTRON KINETIC PROBLEM

Analysis of ADS may require spatial and spectral neutron kinetics

Multidimensional evaluations require large computational effort for a direct numerical solution

$\Longrightarrow$ Quasi-statics is an attracting method

The subcriticality of ADS requires the development of ad-hoc numerical methods

Classic **Quasi-Static Method**

Steady-state problem (source-free critical system, reference reactor)

$$(\hat{\mathrm{L}}_0+\hat{\mathrm{M}}_0)N_0(r,E,\Omega)=0, \qquad N_0(r_S,E,\Omega_{in})=0$$

steady-state multiplication operator

$$\hat{\mathrm{M}}_0=\sum_j\frac{\chi^j(E)}{4\pi}\int dE'\oint d\Omega' v(E')\nu^j(E')\Sigma_f^j(r,E',0)$$

Adjoint problem (source-free critical system, reference reactor)

$$(\hat{\mathrm{L}}_0^+ +\hat{\mathrm{M}}_0^+)N_0^+(r,E,\Omega)=0, \qquad N_0^+(r_S,E,\Omega_{out})=$$

Factorization formula:

$$n(r,E,\Omega,t)=P(t)\varphi(r,E,\Omega;t)$$

* $P(t)$ is the amplitude function;

* $\varphi(r, E, \Omega; t)$ is the shape function;

* Two-scales in time are introduced: the evolution of the amplitude may be much faster than the evolution of the shape;

* As such the factorization is not unique;

Introduce the factorization into the balance equations (shape equations)

$$\begin{cases} P\dfrac{\partial\varphi}{\partial t} + \varphi\dfrac{dP}{dt} = P\hat{\mathrm{B}}\varphi + \displaystyle\sum_{i=1}^{6}\lambda_i\left(\frac{\chi_i}{4\pi}C_i\right) + S \\ \dfrac{\partial(\chi_i C_i/4\pi)}{\partial t} = P\hat{\mathrm{M}}_i\varphi - \lambda_i\left(\dfrac{\chi_i}{4\pi}C_i\right) \end{cases}$$

solve for the delayed neutron precursor concentrations

$$\frac{\chi_i(E)}{4\pi}C_i(\mathbf{r},t) = \frac{\chi_i(E)}{4\pi}C_i(\mathbf{r},t=t_0)e^{-\lambda_i(t-t_0)} + \\ + \int_{t_0}^{t} P(t')\hat{\mathrm{M}}_i\varphi(\mathbf{r},E,\Omega;t')e^{-\lambda_i(t-t')}dt'$$

Project on the solution to the adjoint problem

$$\begin{cases} P\frac{\partial}{\partial t}\left\langle N_0^+ \right| \varphi\rangle + \frac{dP}{dt}\left\langle N_0^+ \right| \varphi\rangle = \\ \qquad P\left\langle N_0^+ \right| \hat{\mathrm{B}}\varphi\right\rangle + \sum_{i=1}^{6}\lambda_i\left\langle N_0^+ \right| \frac{\chi_i}{4\pi}C_i\right\rangle + \left\langle N_0^+ \right| S \\ \frac{\partial}{\partial t}\left\langle N_0^+ \right| \frac{\chi_i}{4\pi}C_i\right\rangle = P\left\langle N_0^+ \right| \hat{\mathrm{M}}_i\varphi\right\rangle - \lambda_i\left\langle N_0^+ \right| \frac{\chi_i}{4\pi}C_i\right\rangle \end{cases}$$

Require a normalization condition for the shape function

$$\frac{\partial}{\partial t}\left\langle N_0^+ \right| \varphi\rangle = 0$$

Definitions:

$$\hat{\mathrm{B}}(t) = \hat{\mathrm{L}}(t) + \hat{\mathrm{M}}_p(t) = (\hat{\mathrm{L}}_0 + \hat{\mathrm{M}}_0) + \delta\hat{\mathrm{B}}(t) - \\ \sum_{i=1}^{6}\hat{\mathrm{M}}_i(0) - \sum_{i=1}^{6}\delta\hat{\mathrm{M}}_i(t) + \sum_{i=1}^{6}\delta\hat{\mathrm{M}}_i(t) = \\ = (\hat{\mathrm{L}}_0 + \hat{\mathrm{M}}_0) + \delta\left[\hat{\mathrm{B}}(t) + \sum_{i=1}^{6}\hat{\mathrm{M}}_i(t)\right] - \sum_{i=1}^{6}\hat{\mathrm{M}}_i(t)$$

Perturbation operator:

$$\delta\hat{\mathrm{K}} = \delta\hat{\mathrm{B}}(t) + \sum_{i=1}^{6} \delta\hat{\mathrm{M}}_i(t)$$

Equations for the amplitude are obtained (**point-like equations**):

$$\begin{cases} \dfrac{dP(t)}{dt} = \dfrac{\rho(t) - \tilde{\beta}}{\Lambda} P(t) + \displaystyle\sum_{i=1}^{6} \lambda_i \tilde{C}_i(t) + \tilde{S}(t) \\ \dfrac{\partial \tilde{C}_i(t)}{\partial t} = \dfrac{\tilde{\beta}_i}{\Lambda} P(t) - \lambda_i \tilde{C}_i(t) \end{cases}$$

Note: if shape is kept constant, the standard point model is obtained.

Kinetic parameters:

$\Rightarrow$ reactivity

$$\rho(t) = \frac{\left\langle N_0^+ \right| \delta\hat{\mathrm{K}}\varphi \Big\rangle}{\left\langle N_0^+ \right| \hat{\mathrm{M}}\varphi \Big\rangle}$$

$\Rightarrow$ effective mean prompt-neutron generation time

$$\Lambda = \frac{\left\langle N_0^+ \right| \varphi \rangle}{\left\langle N_0^+ \right| \hat{\mathrm{M}}\varphi \Big\rangle}$$

$\Rightarrow$ effective delayed neutron fractions

$$\tilde{\beta}_i = \frac{\left\langle N_0^+ \right| \hat{\mathrm{M}}_i\varphi \Big\rangle}{\left\langle N_0^+ \right| \hat{\mathrm{M}}\varphi \Big\rangle}$$

Effective delayed neutron precursor concentrations

$$\tilde{C}_i(t) = \frac{\left\langle N_0^+ \right| \frac{\chi_i}{4\pi} C_i \Big\rangle}{\left\langle N_0^+ \right| \varphi \rangle} = \frac{1}{\Lambda} \frac{\left\langle N_0^+ \right| \frac{\chi_i}{4\pi} C_i \Big\rangle}{\left\langle N_0^+ \right| \hat{\mathrm{M}}\varphi \Big\rangle}$$

Effective external source

$$\tilde{S}(t) = \frac{\left\langle N_0^{+} \middle| S \right\rangle}{\left\langle N_0^{+} \middle| \varphi \right\rangle} = \frac{1}{\Lambda} \frac{\left\langle N_0^{+} \middle| S \right\rangle}{\left\langle N_0^{+} \middle| \hat{\mathrm{M}}\varphi \right\rangle}$$

For the numerical solution of the problem two time intervals are introduced:

$\Rightarrow$ Shape interval (slow phenomena) Δt_{φ}
$\Rightarrow$ Amplitude interval (fast phenomena) Δt_P

Discrete equations for the the shape:

$$T = t_0 + \Delta t_\varphi$$

$$P(T)\frac{\varphi(T) - \varphi(t_0)}{\Delta t_\varphi} + \varphi(T)\dot{P}(T) = P(T)\hat{\mathrm{B}}\varphi(T) +$$

$$\sum_{i=1}^{6} \lambda_i \left[\frac{\chi_i(E)}{4\pi} C_i(t_0) e^{-\lambda_i \Delta t_\varphi} + \right.$$

$$\left. \int_{t_0}^{T} P(t')\hat{\mathrm{M}}_i \varphi(t_0) e^{-\lambda_i (T - t')} dt' + S(T) \right]$$

$$\Longrightarrow \varphi(T)$$

In general, $\varphi(T)$ will not satisfy the required normalization condition, namely:

$$\gamma(T) = \left\langle N_0^+ \right| \varphi(T)\rangle \neq \gamma(t_0) = \left\langle N_0^+ \right| \varphi(t_0)\rangle$$

An iterative process is necessary.

$$P(T)\frac{\varphi^{(n)}(T)-\varphi(t_0)}{\Delta t_\varphi}+\varphi^{(n)}(T)\dot{P}^{(n)}(T)=$$

$$P(T)\hat{\mathrm{B}}\varphi^{(n)}(T)+\sum_{i=1}^{6}\lambda_i\left[\frac{\chi_i(E)}{4\pi}C_i(t_0)e^{-\lambda_i\Delta t_\varphi}\right.$$

$$\left.+\int_{t_0}^{T}P(t')\hat{\mathrm{M}}_i\varphi(t_0)e^{-\lambda_i(T-t')}dt'+S(T)\right]$$

$$\Longrightarrow \varphi^{(n)}(T)$$

$$\gamma^{(n)}(T)=\left\langle N_0^+\middle|\,\varphi^{(n)}(T)\right\rangle$$

$$\varphi^{(n+1/2)}(T)=\frac{\gamma(t_0)}{\gamma^{(n)}(T)}\varphi^{(n)}(T)$$

The derivative of the amplitude function is allowed to be discontinuous; it is updated according to:

$$\dot{P}^{(n+1)}(T) = P(T)\frac{\left\langle N_0^+ \right| \hat{\mathrm{B}}\varphi^{(n+1/2)} \Big\rangle}{\left\langle N_0^+ \right| \varphi^{(n+1/2)}(T) \Big\rangle} + \sum_{i=1}^{6} \lambda_i \frac{\left\langle N_0^+ \right| \frac{\chi_i}{4\pi} C_i \Big\rangle}{\left\langle N_0^+ \right| \varphi^{(n+1/2)}(T) \Big\rangle} + \frac{\left\langle N_0^+ \right| S(T) \rangle}{\left\langle N_0^+ \right| \varphi^{(n+1/2)}(T) \Big\rangle}$$

Source-Driven Problems:
Quasi-static needs to be adapted

The reference reactor is driven by an external source $\Rightarrow$ the initial shape is assumed as solution of the steady-state equation:

$$(\hat{\mathrm{L}}_0 + \hat{\mathrm{M}}_0) N_0(r, E, \Omega) + S_0 = 0,$$

$$N_0(r_S, E, \Omega_{in}) = 0$$

How to define the adjoint?

a) introduce a multiplication eigenvalue in the adjoint equation

$$(\hat{\mathrm{L}}_0^+ + \frac{1}{k_0}\hat{\mathrm{M}}_0^+)N_{0,cr}^+(r, E, \Omega) = 0,$$

$$N_{0,cr}^+(r_S, E, \Omega_{out}) = 0$$

b) introduce an adjoint source

$$(\hat{\mathrm{L}}_0^+ + \hat{\mathrm{M}}_0^+)N_{0,s}^+(r, E, \Omega) + S^+ = 0,$$

$$N_{0,s}^+(r_S, E, \Omega_{out}) = 0$$

Problem: how to define the adjoint source?

Possible solution:

$\Rightarrow$ if importance is defined as the number of fission neutrons to be produced per neutron injected at a point in phase space (variational interpretation), the adjoint source is

$$S^{+} = \nu\Sigma_{f}$$

Consequence on the reactivity of the system:

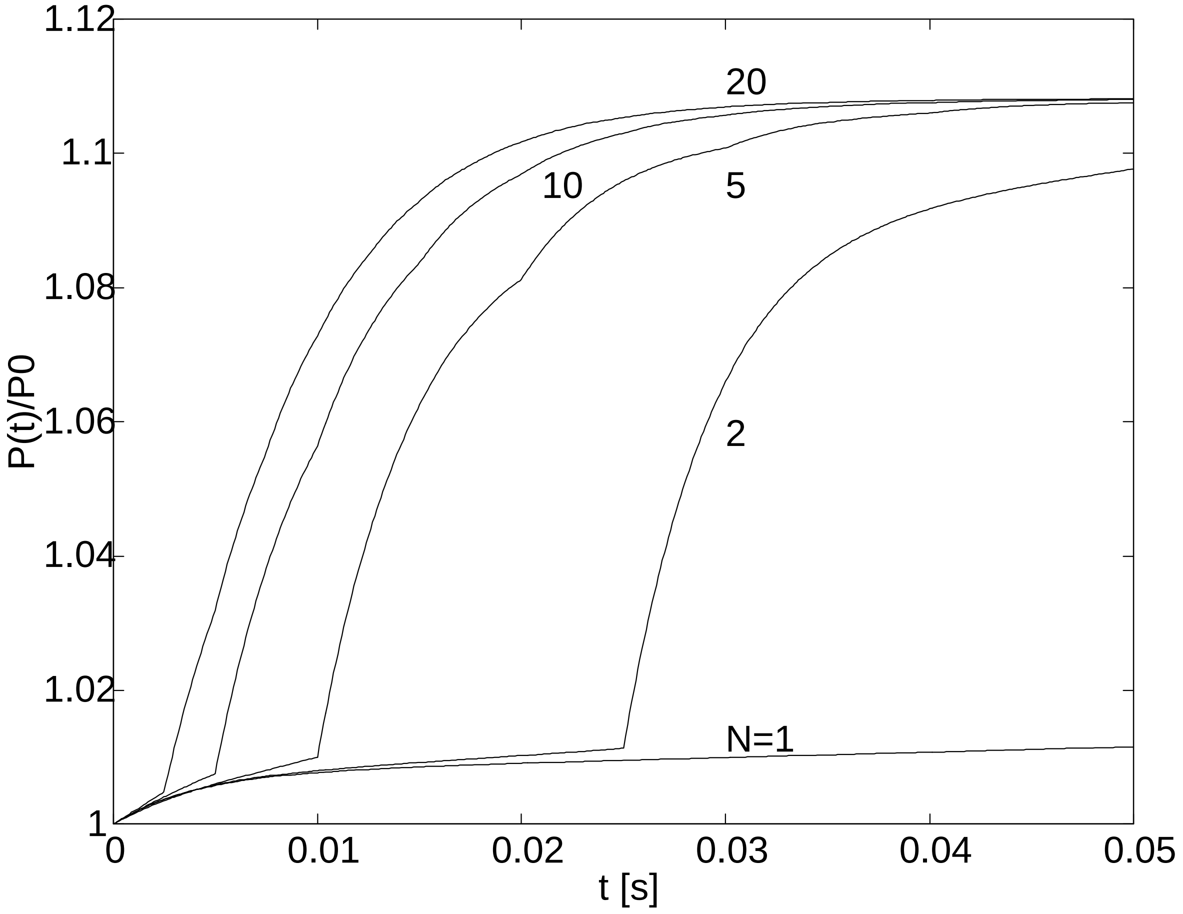
$$\rho = \rho_0 + \rho_p$$

ρ_p is the pertubation reactivity

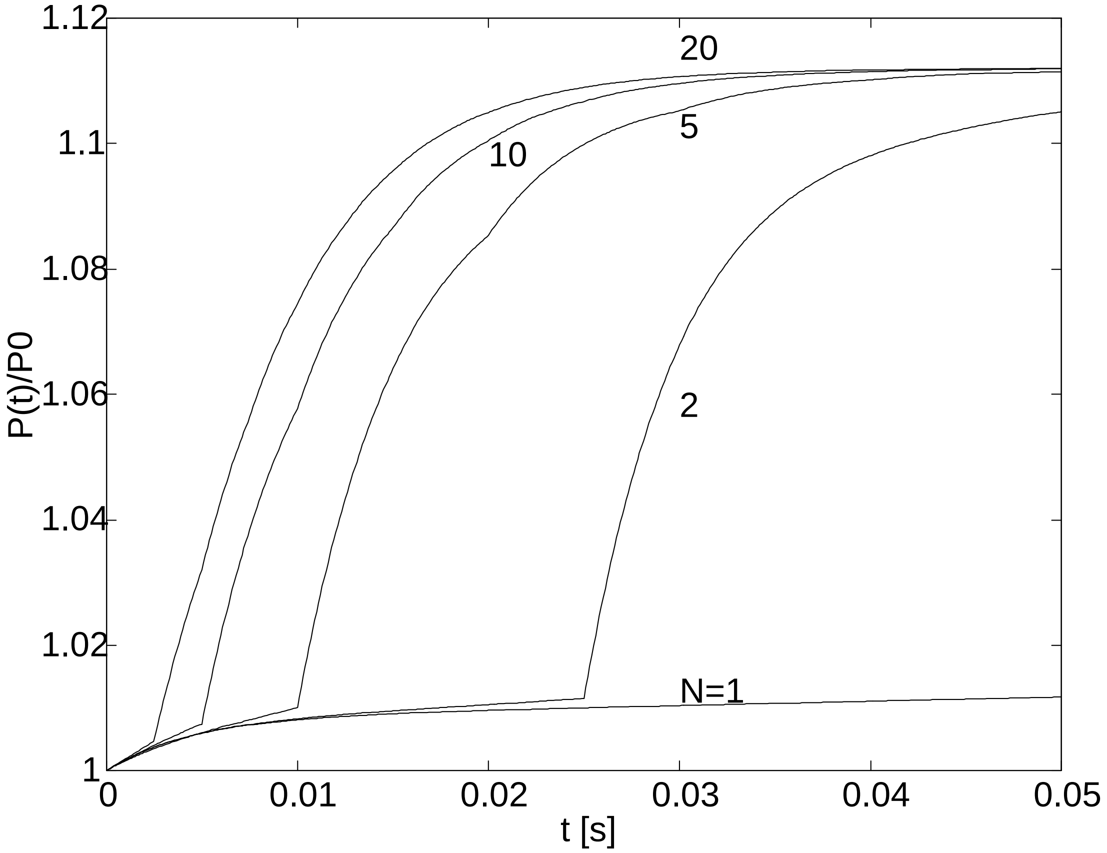
ρ_0 is connected to the initial subcriticality level

$$\rho_0 = \frac{\left\langle N^{+}_{0,cr} \middle| \hat{\mathrm{M}}_0 \varphi \right\rangle}{\left\langle N^{+}_{0,cr} \middle| \hat{\mathrm{M}} \varphi \right\rangle} \left(\frac{k-1}{k} \right)$$

$$\rho_0 = -\frac{\left\langle S^{+} | \varphi \right\rangle}{\left\langle N^{+}_{0,s} \middle| \hat{\mathrm{M}} \varphi \right\rangle}$$



Power: weight function as source adjoint.



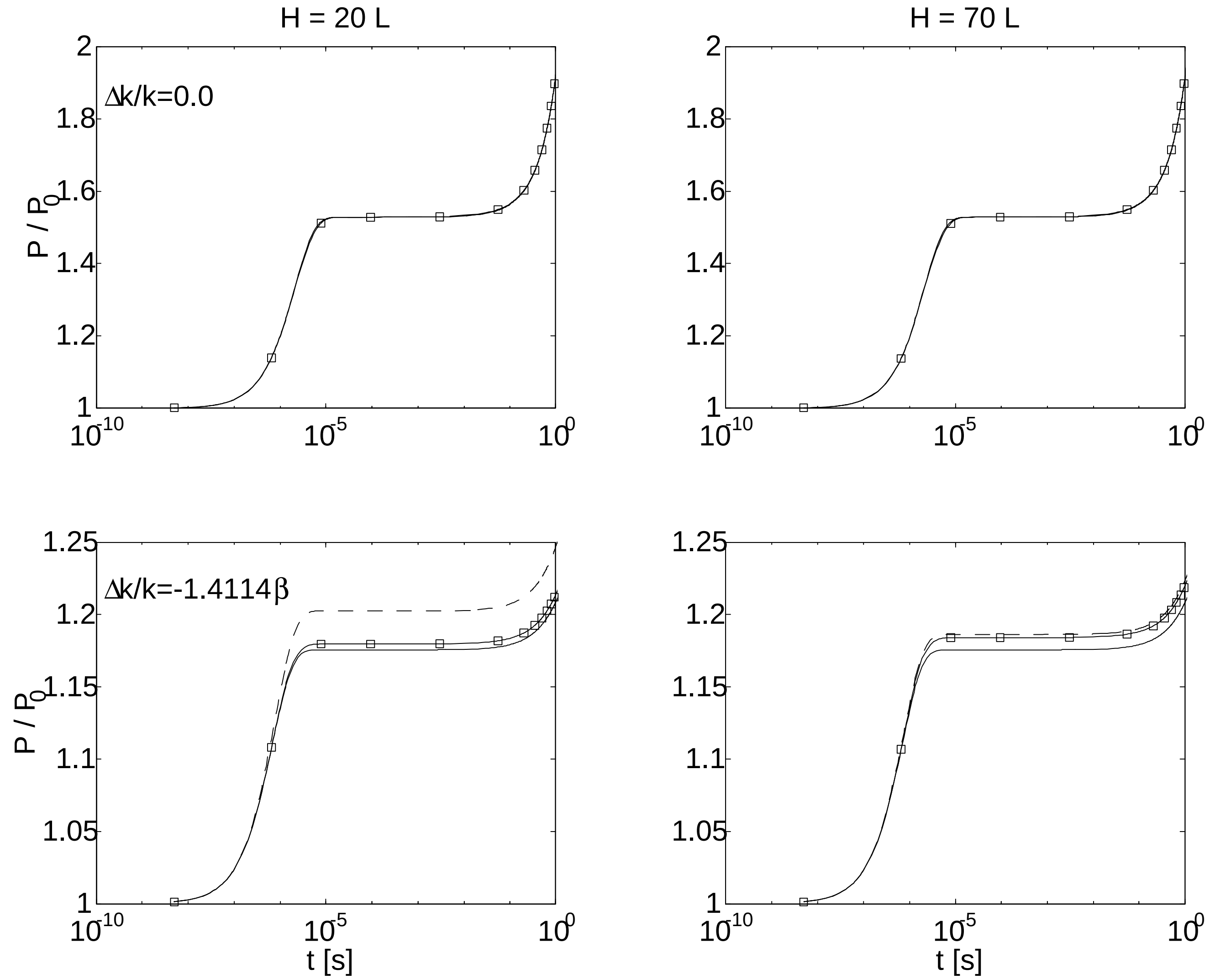
Power: weight function as critical adjoint.

DISCUSSION ON THE CHOICE OF THE WEIGHTING FUNCTION IN SEPARATION SCHEMES

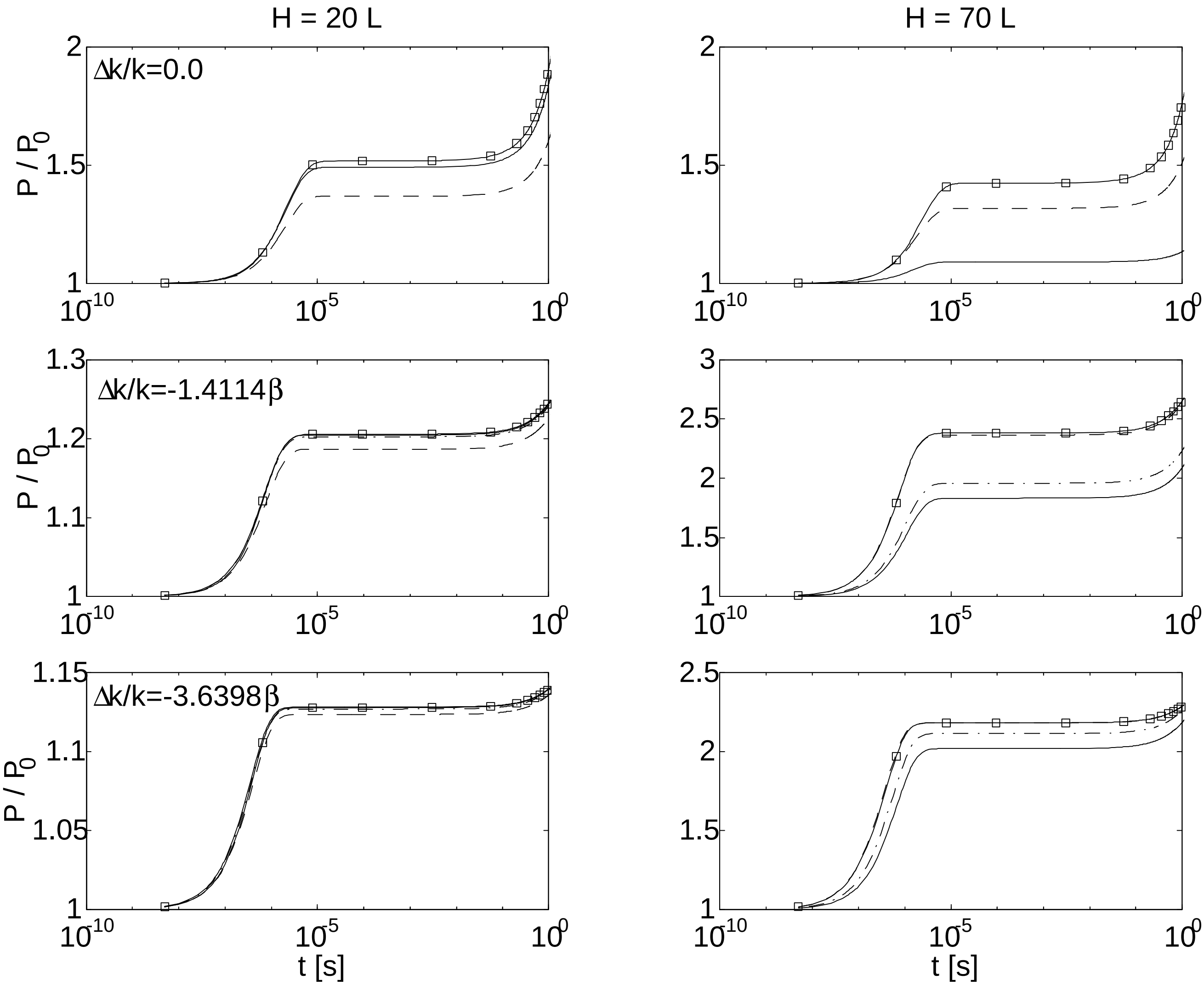
It is an open problem for subcritical systems:

$\Longrightarrow$ Evidence problem, rather then propose solutions;

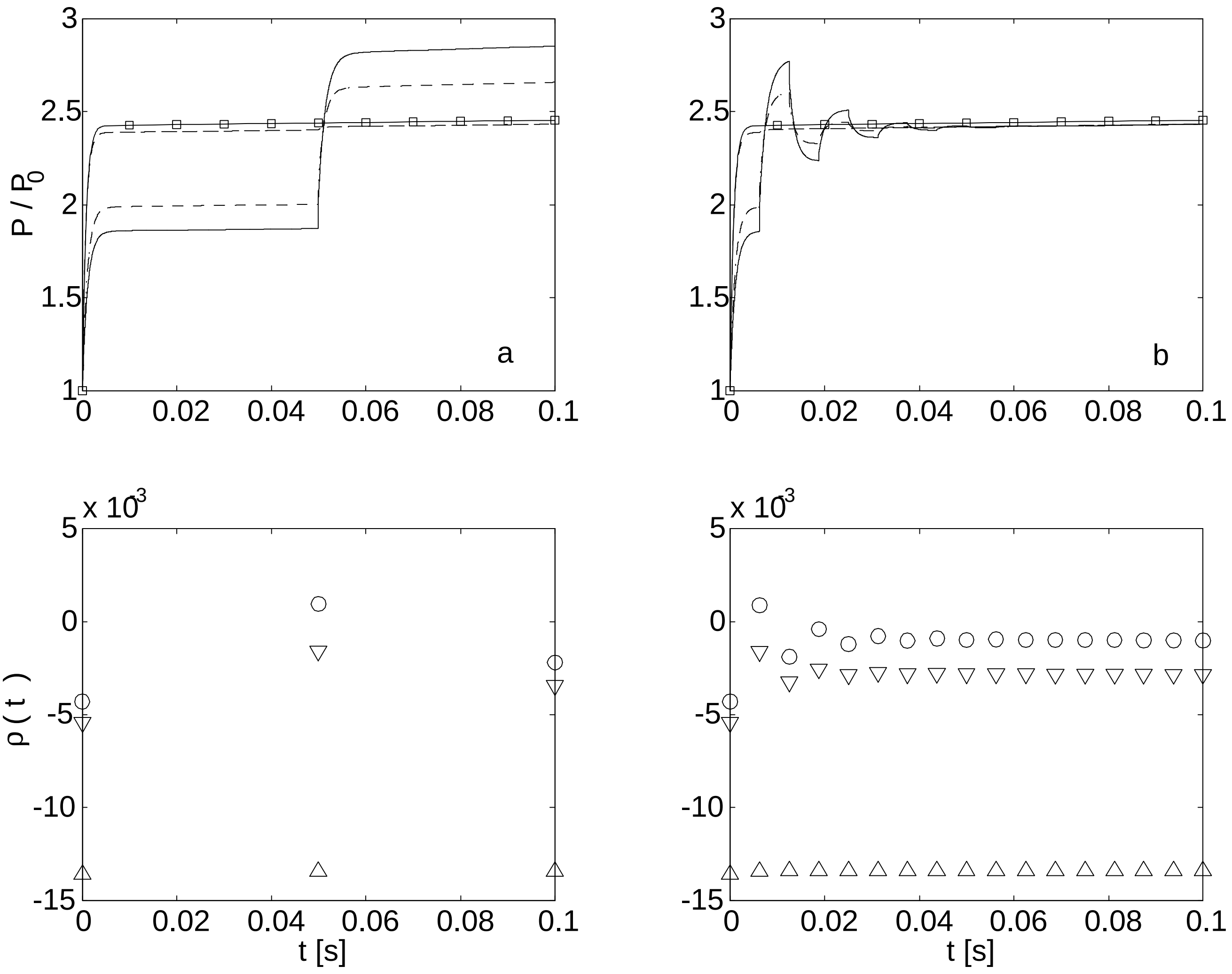
$\Longrightarrow$ Reasonable possibilities: critical adjoint or the adjoint driven by the fission cross-section as a source.



Homogeneous perturbation, systems with different k. Top: initial critical system; bottom: initial subcritical system, $k = 0.98$. Squares: reference; broken line: constant adjoint; solid line: critical adjoint; dot-point line: source-driven adjoint.



Comparison of the power evolutions calculated using different weighting functions in response to a localized perturbation. Squares: reference results; broken line: constant adjoint; solid line: critical adjoint; dot-point line: source-driven adjoint.



Quasi-static calculations with $H = 70L$ and $\Delta k/k = -1.4114\beta$, compared to the full spatial results. One shape update for graphs a), 15 recalculations for case b). Reactivity at the bottom; $\triangledown$: source adjoint; $\triangle$: constant; $\circ$: critical adjoint.

COMPUTATIONAL TOOL

The neutronic module solves multi-group diffusion equations in cylindrical geometry

The thermal calculation is performed by the channel code TIESTE developed at ENEA (Italy) for accelerator-driven systems cooled by lead-bismuth

The instants for the thermal calculation are chosen according to the power change of the system

The average values of the power are computed for pre-defined zones of the multiplying structure and input into the thermal module

The thermal code can represent a single reactor channel with fuel, clad and coolant

At the end of the thermal calculation, the cross-sections are updated according to a linear interpolation process between tables of data generated at different fuel and coolant temperatures

DEVELOPMENTS

The quasi-static method proves to be a powerful and computationally efficient tool for the analysis of source-driven systems

Accurate predictions of the transient behavior in non-linear conditions dominated by thermal feed-back can be attained

Development:

$\Longrightarrow$ An improvement: the **multipoint** method

Balance equations in discretized form:

$$\begin{cases} \dfrac{1}{v_m}\dfrac{d\phi_{nm}}{dt} = \sum_{n'}\sum_{m'} k_{nm,n'm'}\phi_{n'm'} + \\ \qquad \sum_{i=1}^{6} \lambda_i \chi_{i,m} C_{i,n} + S_{nm} \\ \dfrac{dC_{i,n}}{dt} = \beta_i \sum_{m'} f_{nm'}\phi_{nm'} - \lambda_i C_{i,n} \qquad i = 1, 2, ..., 6 \end{cases}$$

$$\phi_{nm}(t) = \phi(\mathbf{r}_n, V_m, t) \qquad C_{i,n}(t) = C_i(\mathbf{r}_n, t)$$

$$\phi_{nm}(t) = A_{NM}(t)\varphi_{nm}(t) \qquad \mathbf{r}_n, V_m \in \Gamma_{NM}$$

definition of a regionwise inner product

$$\langle w|\, g\rangle = \left[\sum_n \sum_m\right]_{NM} w_{nm} g_{nm}$$

Introduce factorization into the balance equations:

$$\begin{cases} \frac{1}{v_m}\varphi_{nm}\frac{dA_{NM}}{dt} + \frac{1}{v_m}A_{NM}\frac{d\varphi_{nm}}{dt} = \\ \sum_{N'}\sum_{M'}\left[\sum_{n'}\sum_{m'}\right]_{N'M'} k_{nm,n'm'}\varphi_{n'm'}A_{N'M'} + \\ \qquad \sum_{i=1}^{6}\lambda_i\chi_{i,m}C_{i,n} + S_{nm} \\ \\ \frac{dC_{i,n}}{dt} = \beta_i\sum_{M'}\left[\sum_{m'}\right]_{M'} f_{nm'}\varphi_{nm'}A_{NM'} - \lambda_i C_{i,n} \\ \qquad i = 1, 2, ..., 6, \quad \mathbf{r}_n, V_m \in \Gamma_{NM} \end{cases}$$

Projection: multiply by w_{nm} and sum on NM

Normalization condition

$$\frac{d}{dt}\left[\sum_{n}\sum_{m}\right]_{NM} w_{nm}\frac{1}{v_m}\varphi_{nm}(t) = \frac{d}{dt}\gamma_{NM} = 0$$

Point-to-point transfer term

$$\left[\sum_{n}\sum_{m}\right]_{NM} w_{nm}\sum_{N'}\sum_{M'}\left[\sum_{n'}\sum_{m'}\right]_{N'M'} \ldots$$

$$k_{nm,n'm'}\varphi_{n'm'}A_{N'M'} =$$

$$\sum_{N'}\sum_{M'}\left[\sum_{n}\sum_{m}\right]_{NM} \ldots$$

$$\left(w_{nm}\left[\sum_{n'}\sum_{m'}\right]_{N'M'} k_{nm,n'm'}\varphi_{n'm'}\right)A_{N'M'}$$

$$\sum_{N'}\sum_{M'}K^{*}_{NM,N'M'}A_{N'M'}$$

Multipoint equations:

$$\begin{cases} \dfrac{dA_{NM}}{dt} = \displaystyle\sum_{N'}\sum_{M'} K_{NM,N'M'}A_{N'M'} + \\ \qquad\qquad \displaystyle\sum_{i=1}^{6}\lambda_i C_{i,NM} + S_{NM} \\ \dfrac{dC_{i,NM}}{dt} = \beta_i \displaystyle\sum_{M'} F_{i,NM,M'}A_{NM'} - \lambda_i C_{i,NM} \\ \qquad i = 1, 2, ..., 6, \end{cases}$$

point-to-point coupling coefficients

$$K_{NM,N'M'} = \frac{1}{\gamma_{NM}} \left[\sum_{n}\sum_{m}\right]_{NM} \Bigg(w_{nm} \left[\sum_{n'}\sum_{m'}\right]_{N'M'} k_{nm,n'm'}\varphi_{n}$$

effective multipoint source

$$S_{NM} = \frac{1}{\gamma_{NM}} \left[\sum_{n}\sum_{m}\right]_{NM} w_{nm}S_{nm}$$

effective multipoint delayed precursor concentrations

$$C_{i,NM} = \frac{1}{\gamma_{NM}} \left[\sum_{n}\sum_{m}\right]_{NM} w_{nm}\chi_{i,m}C_{i,n}$$

delayed neutron production coefficients

$$F_{i,NM,M'} = \frac{1}{\gamma_{NM}} \left[\sum_{n}\sum_{m}\right]_{NM} w_{nm}\chi_{i,m}\beta_{i} \left[\sum_{m'}\right]_{M'} f_{nm'}\varphi_{nm'}$$

Multipoint can be included into a quasi-static scheme

Solution of the ”slow” shape equation

$$T = t_0 + \Delta t_\varphi$$

$$\frac{1}{v_m}\dot{A}_{NM}(T)\varphi_{nm}(T) + \frac{1}{v_m}A_{NM}(T)\frac{\varphi_{nm}(T) - \varphi_{nm}(t}{\Delta t_\varphi}$$

$$\sum_{N'}\sum_{M'}\left[\sum_{n'}\sum_{m'}\right]_{N'M'} k_{nm,n'm'}(T)\varphi_{n'm'}(T)A_{N'M'}(T$$

$$+\sum_{i=1}^{6}\lambda_i\chi_{i,m}C_{i,n}(T) + S_{nm}(T)$$

$$C_{i,n}(T) = C_{i,n}(t_0)e^{-\lambda_i\Delta t_\varphi}+$$

$$\int_{t_0}^{T}\beta_i\sum_{M'}\left[\sum_{m'}\right]_{M'} f_{nm'}(T)\varphi_{nm'}(t_0)A_{N'M'}(t')\times$$

$$e^{-\lambda_i(T-t')}dt'$$

Iteration scheme

$$\frac{1}{v_m}\dot{A}^{(l)}_{NM}(T)\varphi^{(l)}_{nm}(T)+\frac{1}{v_m}A_{NM}(T)\frac{\varphi^{(l)}_{nm}(T)-\varphi_{nm}(t}{\Delta t_{\varphi}}$$

$$\sum_{N'}\sum_{M'}\left[\sum_{n'}\sum_{m'}\right]_{N'M'}k_{nm,n'm'}(T)\varphi^{(l)}_{n'm'}(T)A_{N'M'}($$

$$\sum_{i=1}^{6}\lambda_i\chi_{i,m}C_{i,n}(T)+S_{nm}(T)$$

$$\gamma^{(l)}_{NM}(T)=\left[\sum_{n}\sum_{m}\right]_{NM}w_{nm}\frac{1}{v_m}\varphi^{(l)}_{nm}(T)$$

$$\varphi^{(l+1/2)}_{nm}(T)=\frac{\varphi^{(l)}_{nm}(T)}{\gamma^{(l)}_{NM}(T)}\gamma_{NM}(t_0)$$

$$\dot{A}^{(l+1)}_{NM}(T) = \gamma^{(l+1/2)}_{NM}(T) \sum_{N'} \sum_{M'} \left[\sum_{n} \sum_{m} \right]_{NM} \cdots$$

$$\left(w_{nm} \left[\sum_{n'} \sum_{m'} \right]_{N'M'} k_{nm,n'm'}(T) \varphi^{(l+1/2)}_{n'm'}(T) \right) A_{N'.}$$

$$+ \frac{\left[\sum_{n} \sum_{m} \right]_{NM} w_{nm} \chi_{i,m} C_{i,n}(T)}{\gamma^{(l+1/2)}_{NM}(T)} +$$

$$\frac{\left[\sum_{n} \sum_{m} \right]_{NM} w_{nm} S_{nm}(T)}{\gamma^{(l+1/2)}_{NM}(T)}$$

The Adiabatic Scheme

Use as equation for the determination of the shape a static equation with an eigenvalue (homogeneous problem):

$$(\hat{\mathrm{L}}(t) + \frac{1}{k(T)}\hat{\mathrm{M}}(T))N_k(r, E, \Omega; T) = 0,$$

$$N_k(r_S, E, \Omega_{in}) = 0$$

normalization condition:

$$\left\langle N_0^+ \right| N_k(T) \rangle = \left\langle N_0^+ \right| N_0 \rangle$$

reactivity takes the usual static form:

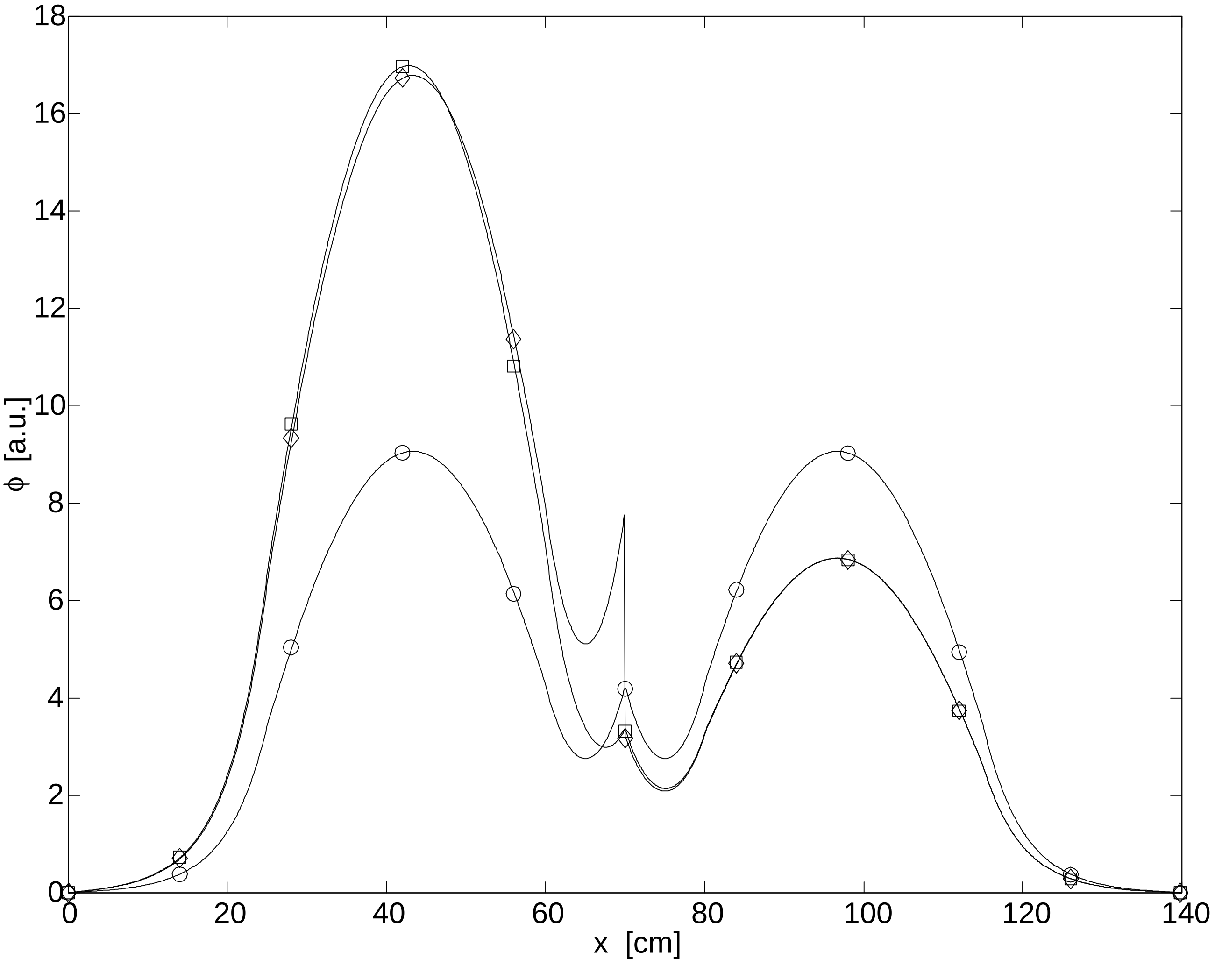
$$\rho(T) = \frac{k(T) - 1}{k(T)}$$

Concluding remarks

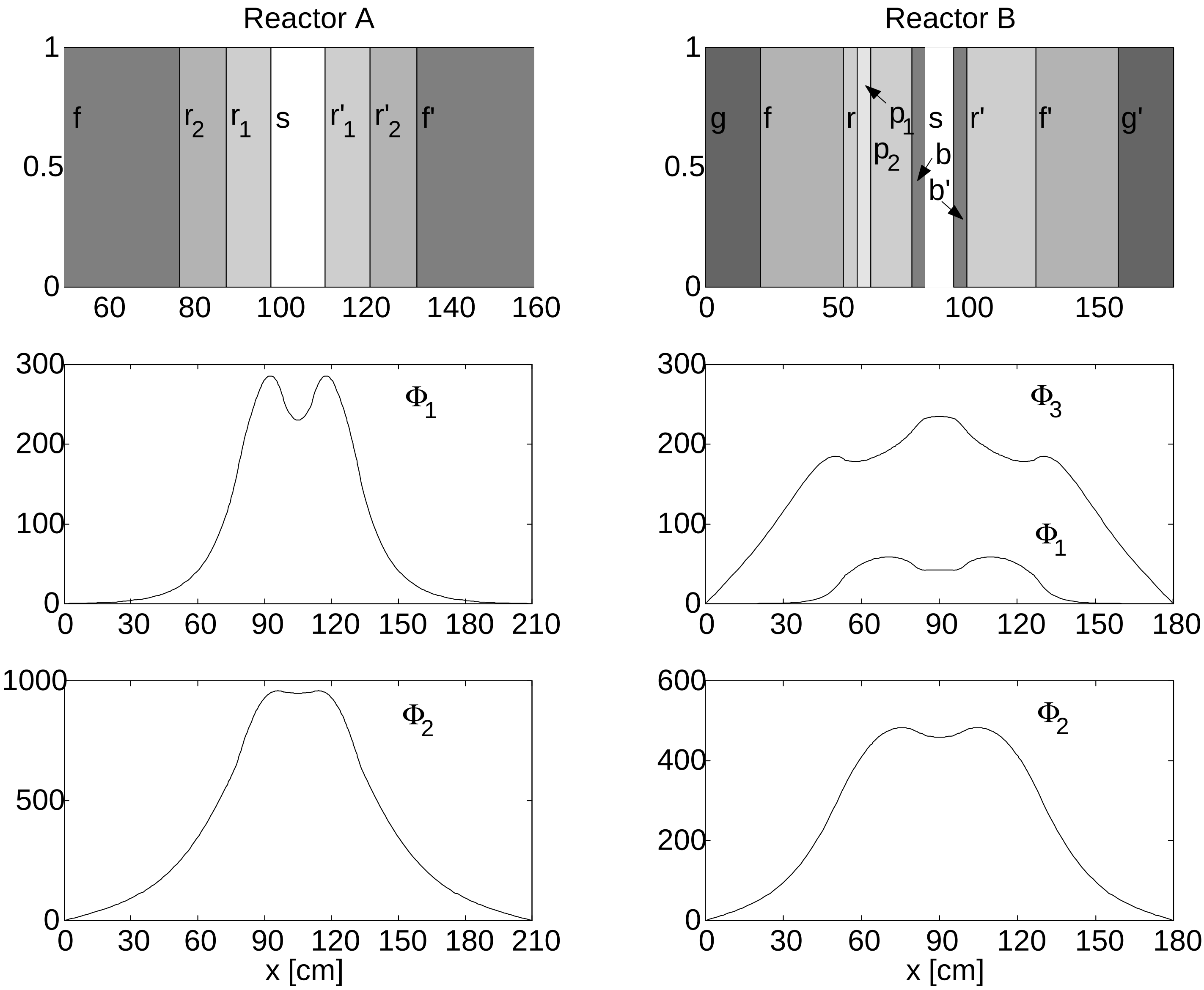
- multipoint is effective in many reactor kinetics problems

- the method can easily be included within quasi-statics, greatly enhancing its performance

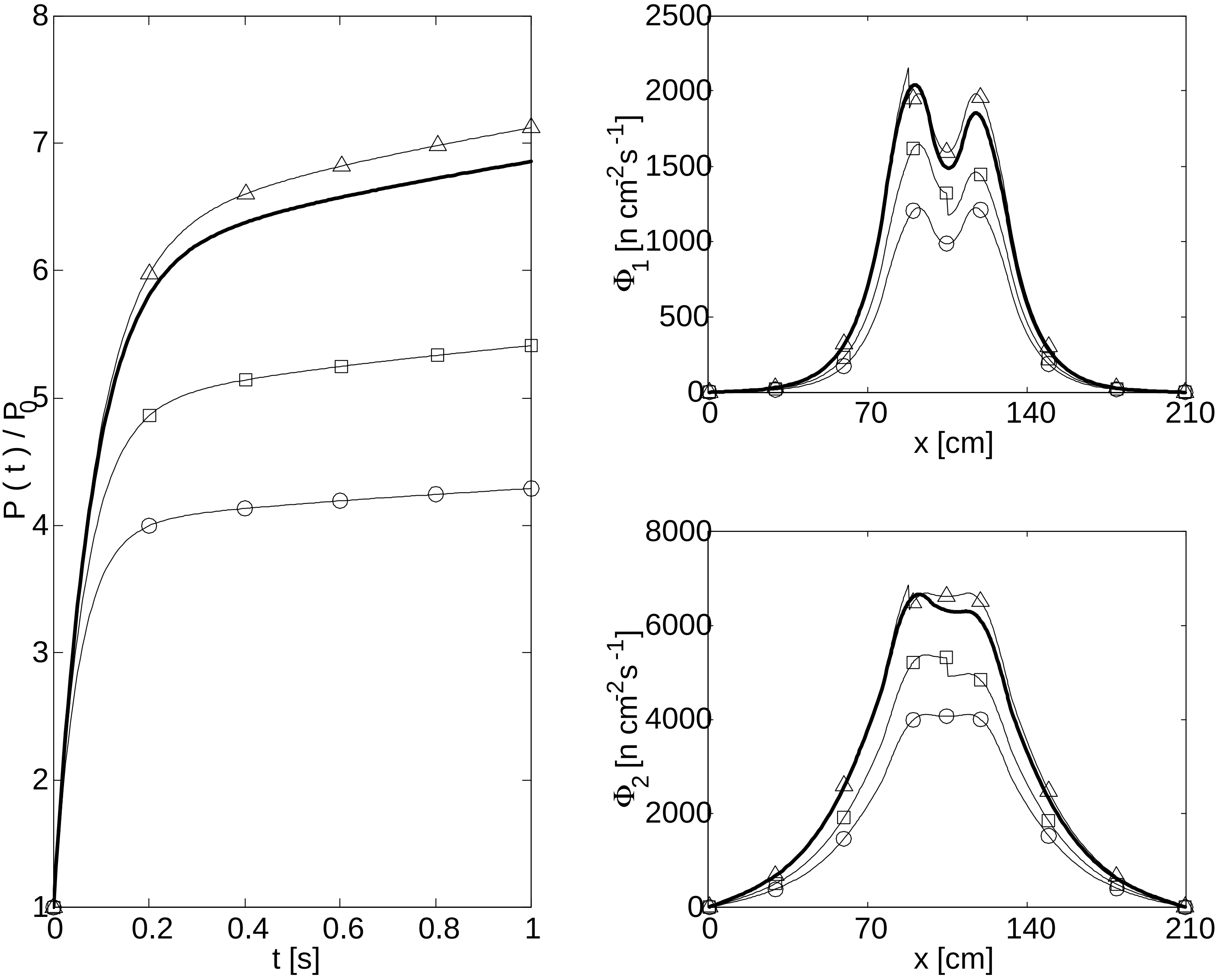
- development: apply multipoint to angular schemes in transport calculations



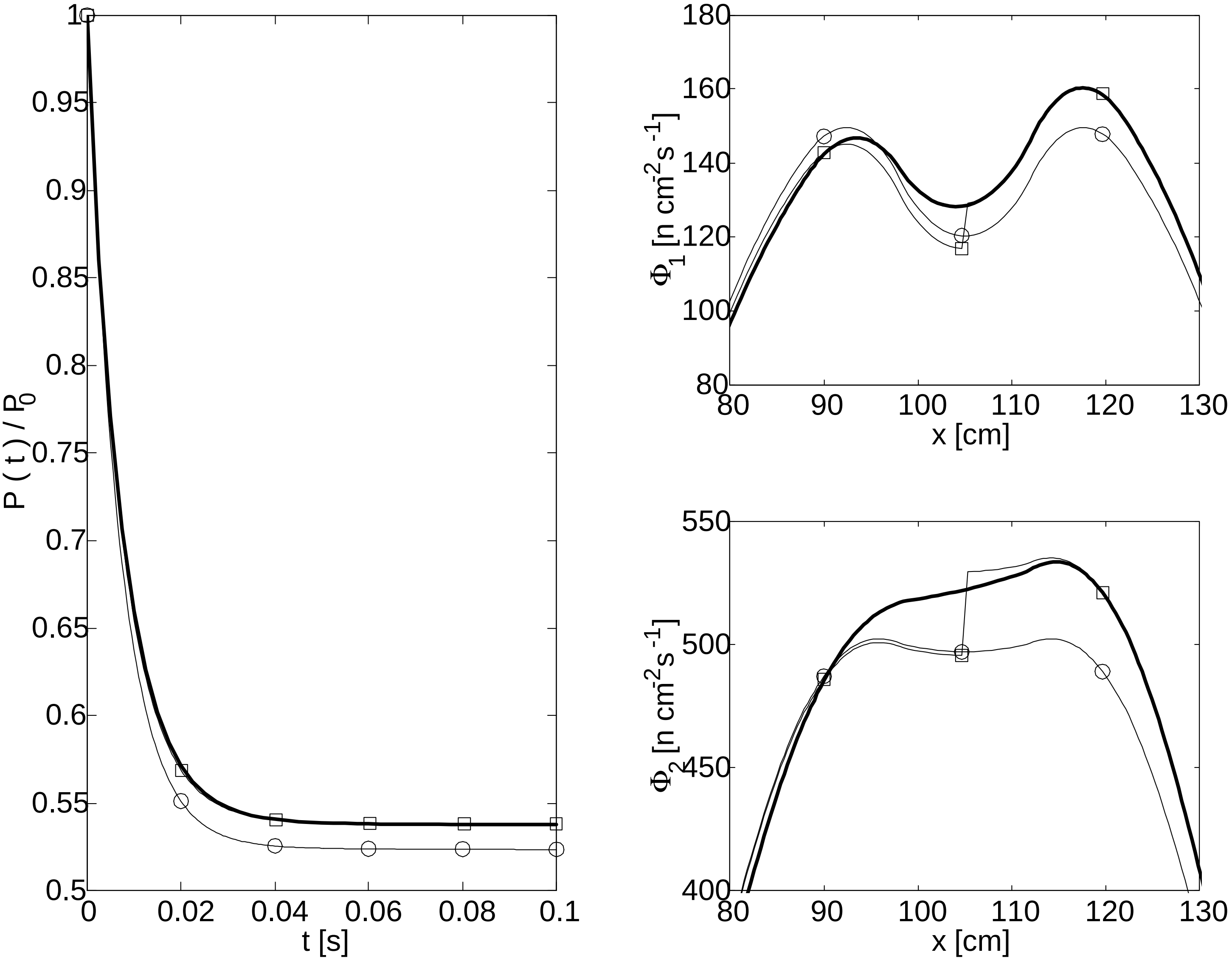
1.Flux at end of transient: ○ : point kinetics; ◇ : 2-point kinetics; □ : exact solution.



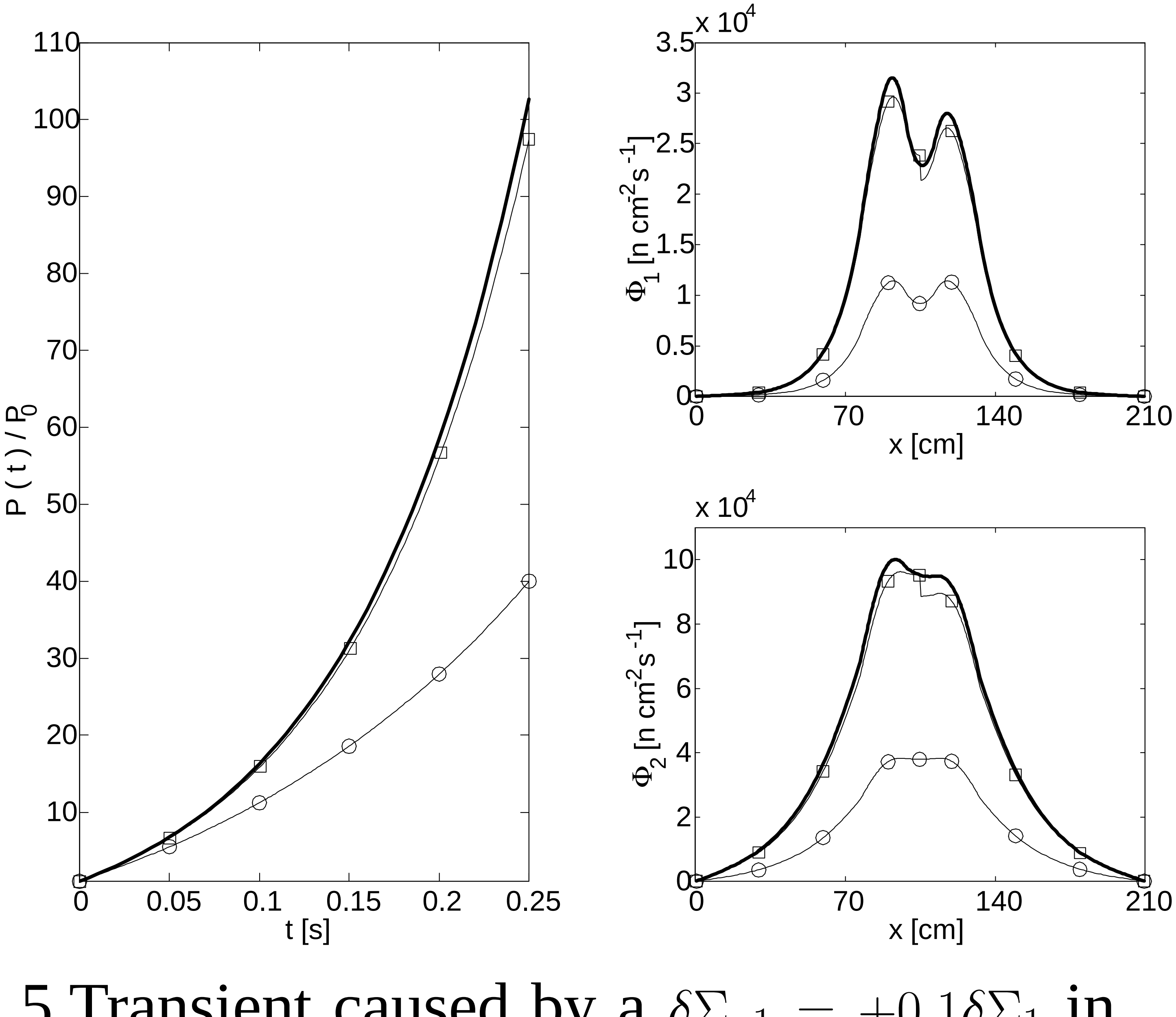
2.Material configurations. s: source channel; r, r’: multiplying zones, Masurca type; r_1, r_1’: multiplying zones, Myrrha type (low enrichment); r_2, r_2’: multiplying zone, Myrrha type (high enrichment); f, f’: reflectors; b,b’: lead buffers; p_1, p_2: perturbed zones; g, g’: shields. Initially, r, p_1 and p_2 have the same properties as r’. Flux in steady state, 2-group diffusion for A and 3-group diffusion for B



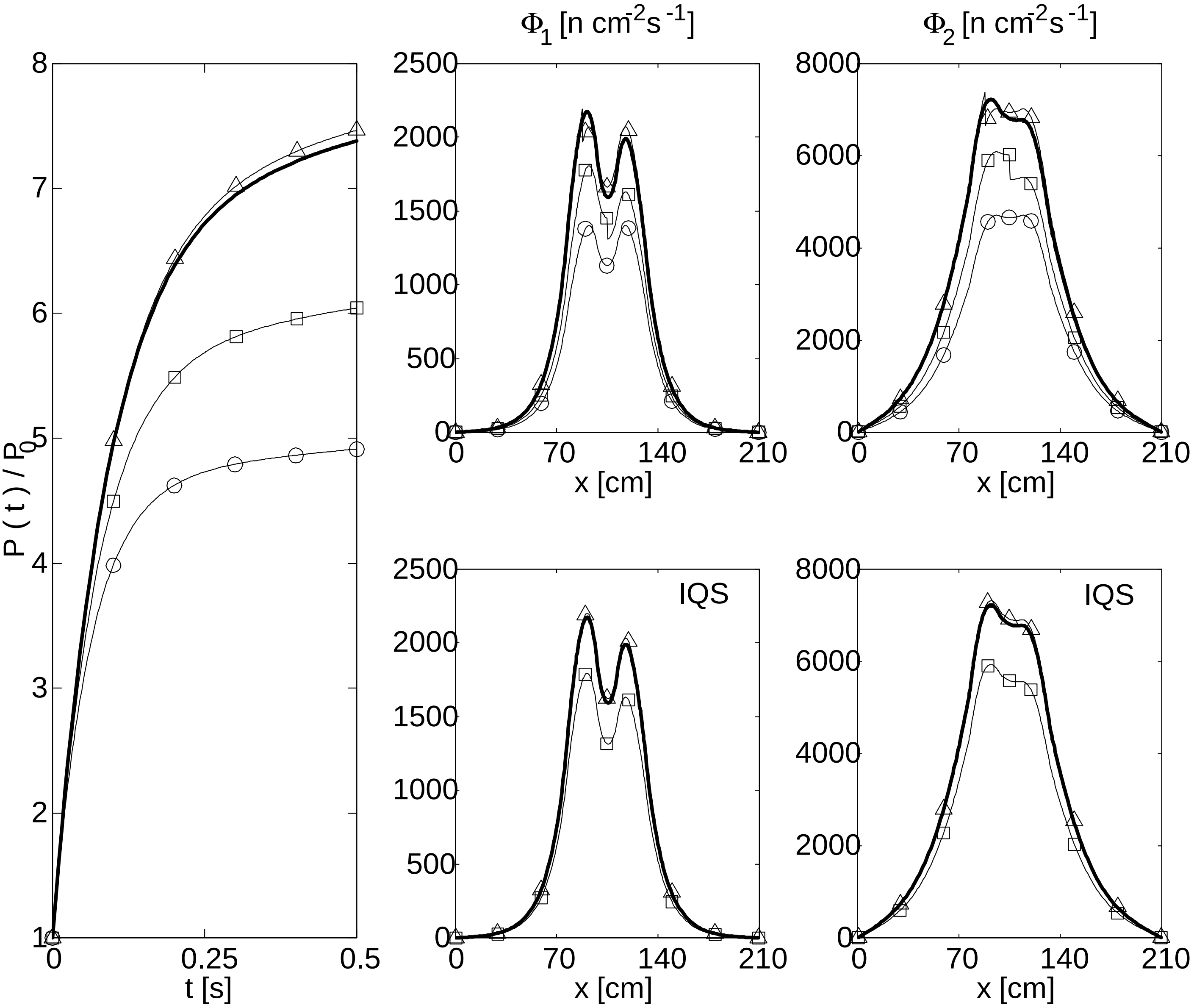
3.Transient caused by a $\delta\Sigma_{a1} = -0.1\delta\Sigma_1$ in the r$_2$ zone of system A. The multiplication constant becomes $k_{eff} = 0.998447$. On the right, the fluxes at the last time considered in the transient ($t = 1$ s) are plotted.



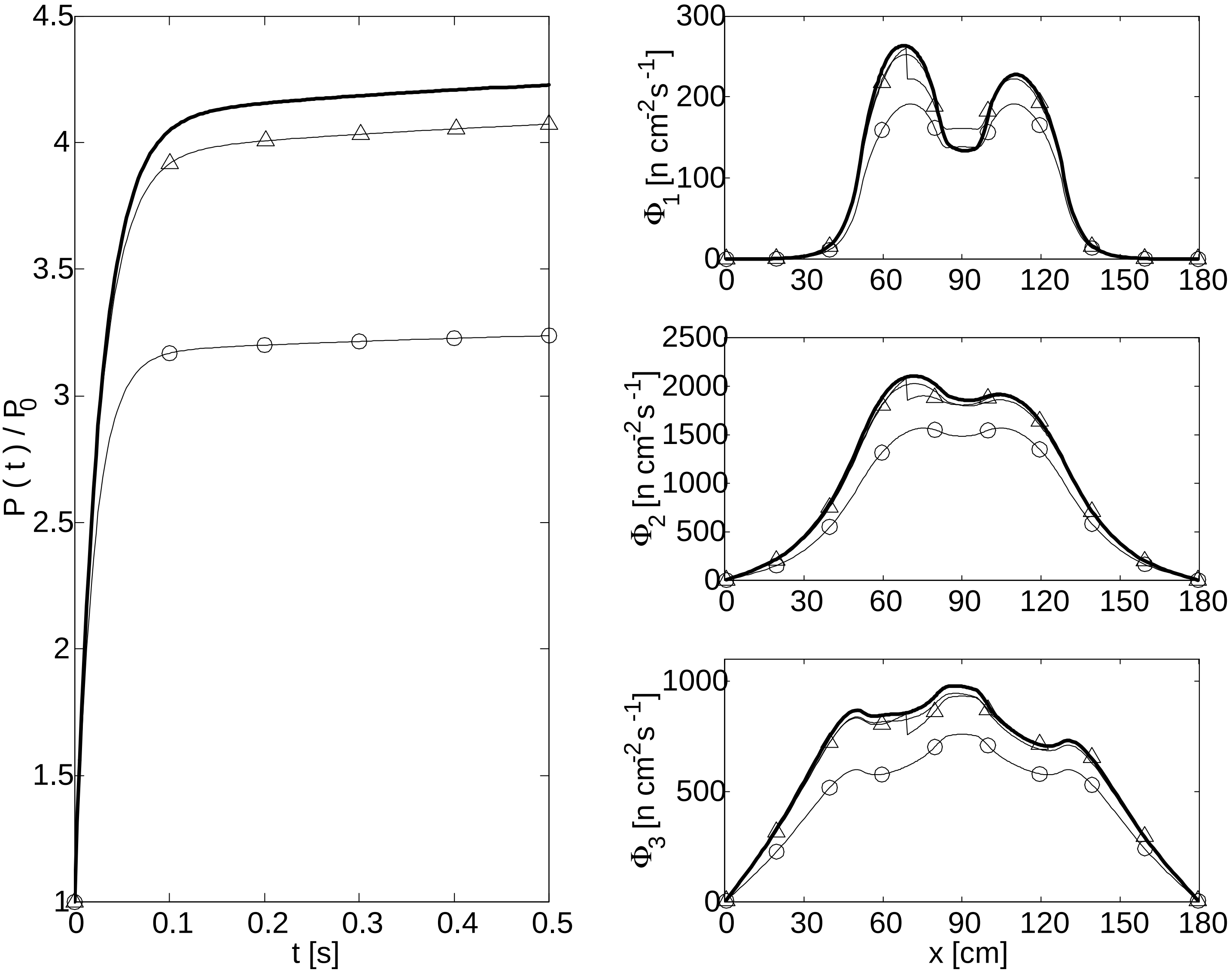
4.Transient caused by a $\delta\Sigma_{a1} = +0.05\delta\Sigma_1$ in the r$_1$ and r$_2$ zones of system A. The multiplication constant becomes $k_{eff} = 0.961767$. Final fluxes on the right.



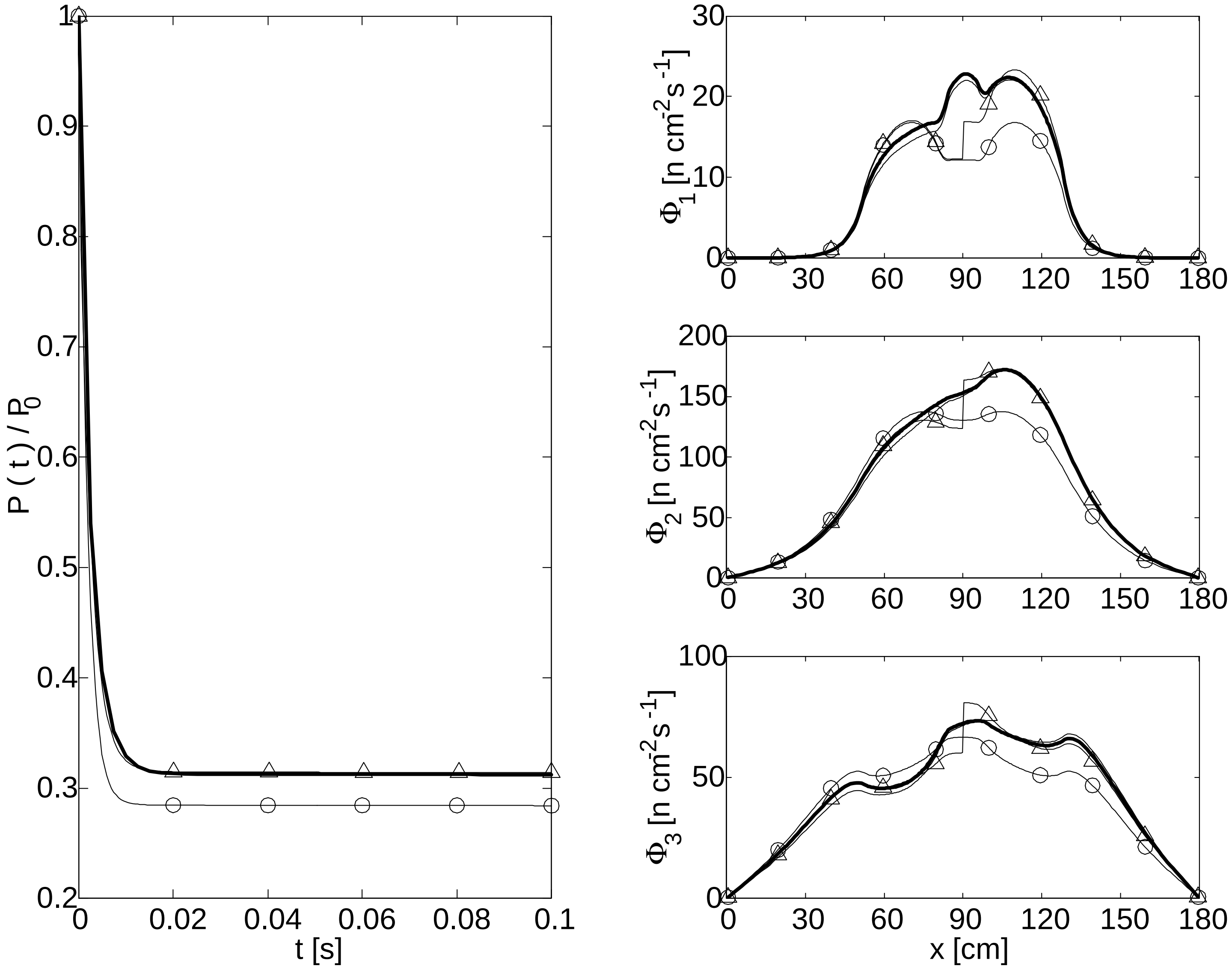
5.Transient caused by a $\delta\Sigma_{a1} = +0.1\delta\Sigma_1$ in the r$_1$ zone of system A. The multiplication constant becomes $k_{eff} = 1.00489$.



6.Transient caused by a $\delta\Sigma_{a1} = -0.05\delta\Sigma_1$ and $\delta\Sigma_{a2} = -0.1\delta\Sigma_2$ in the r$_2$ zone of system A. The multiplication constant becomes $k_{eff} = 0.998850$. In the right on the top, comparison of reference, point and two-point kinetics can be seen. At the bottom, IQS results are reported.



7.Transient caused by a $\delta\Sigma_{ag} = -0.2\delta\Sigma_g$ for all groups in the p$_1$ zone of system B. The multiplication constant becomes $k_{eff} = 0.994264$.



8.Transient caused by a $\delta\Sigma_{ag} = +0.2\delta\Sigma_g$ for all groups in the p$_2$ zone of system B. The final multiplication constant is $k_{eff} = 0.912024$.