

# THE GENERAL NEUTRON KINETIC PROBLEM

Analysis of ADS may require spatial and spectral neutron kinetics

Multidimensional evaluations require large computational effort for a direct numerical solution

⇒ Quasi-statics is an attracting method

The subcriticality of ADS requires the development of ad-hoc numerical methods

## Classic Quasi-Static Method

Steady-state problem (source-free critical system, reference reactor)

$$(\hat{L}_0 + \hat{M}_0)N_0(r, E, \Omega) = 0, \quad N_0(r_S, E, \Omega_{in}) = 0$$

steady-state multiplication operator

$$\hat{M}_0 = \sum_j \frac{\chi^j(E)}{4\pi} \int dE' \oint d\Omega' v(E') \nu^j(E') \Sigma_f^j(r, E', 0)$$

Adjoint problem (source-free critical system, reference reactor)

$$(\hat{L}_0^+ + \hat{M}_0^+)N_0^+(r, E, \Omega) = 0, \quad N_0^+(r_S, E, \Omega_{out}) = 0$$

Factorization formula:

$$n(r, E, \Omega, t) = P(t)\varphi(r, E, \Omega; t)$$

- \*  $P(t)$  is the amplitude function;
- \*  $\varphi(r, E, \Omega; t)$  is the shape function;
- \* Two-scales in time are introduced: the evolution of the amplitude may be much faster than the evolution of the shape;
- \* As such the factorization is not unique;

Introduce the factorization into the balance equations (shape equations)

$$\begin{cases} P \frac{\partial \varphi}{\partial t} + \varphi \frac{dP}{dt} = P \hat{B} \varphi + \sum_{i=1}^6 \lambda_i \left( \frac{\chi_i}{4\pi} C_i \right) + S \\ \frac{\partial (\chi_i C_i / 4\pi)}{\partial t} = P \hat{M}_i \varphi - \lambda_i \left( \frac{\chi_i}{4\pi} C_i \right) \end{cases}$$

solve for the delayed neutron precursor concentrations

$$\begin{aligned} \frac{\chi_i(E)}{4\pi} C_i(\mathbf{r}, t) &= \frac{\chi_i(E)}{4\pi} C_i(\mathbf{r}, t = t_0) e^{-\lambda_i(t-t_0)} + \\ &+ \int_{t_0}^t P(t') \hat{M}_i \varphi(\mathbf{r}, E, \Omega; t') e^{-\lambda_i(t-t')} dt' \end{aligned}$$

Project on the solution to the adjoint problem

$$\left\{ \begin{array}{l} P \frac{\partial}{\partial t} \langle N_0^+ | \varphi \rangle + \frac{dP}{dt} \langle N_0^+ | \varphi \rangle = \\ P \langle N_0^+ | \hat{B}\varphi \rangle + \sum_{i=1}^6 \lambda_i \langle N_0^+ | \frac{\chi_i}{4\pi} C_i \rangle + \langle N_0^+ | S \rangle \\ \frac{\partial}{\partial t} \langle N_0^+ | \frac{\chi_i}{4\pi} C_i \rangle = P \langle N_0^+ | \hat{M}_i \varphi \rangle - \lambda_i \langle N_0^+ | \frac{\chi_i}{4\pi} C_i \rangle \end{array} \right.$$

Require a normalization condition for the shape function

$$\frac{\partial}{\partial t} \langle N_0^+ | \varphi \rangle = 0$$

**Definitions:**

$$\begin{aligned} \hat{B}(t) &= \hat{L}(t) + \hat{M}_p(t) = (\hat{L}_0 + \hat{M}_0) + \delta \hat{B}(t) - \\ &\quad \sum_{i=1}^6 \hat{M}_i(0) - \sum_{i=1}^6 \delta \hat{M}_i(t) + \sum_{i=1}^6 \delta \hat{M}_i(t) = \\ &= (\hat{L}_0 + \hat{M}_0) + \delta \left[ \hat{B}(t) + \sum_{i=1}^6 \hat{M}_i(t) \right] - \sum_{i=1}^6 \hat{M}_i(t) \end{aligned}$$

Perturbation operator:

$$\delta\hat{K} = \delta\hat{B}(t) + \sum_{i=1}^6 \delta\hat{M}_i(t)$$

Equations for the amplitude are obtained  
**(point-like equations):**

$$\begin{cases} \frac{dP(t)}{dt} = \frac{\rho(t) - \tilde{\beta}}{\Lambda} P(t) + \sum_{i=1}^6 \lambda_i \tilde{C}_i(t) + \tilde{S}(t) \\ \frac{\partial \tilde{C}_i(t)}{\partial t} = \frac{\tilde{\beta}_i}{\Lambda} P(t) - \lambda_i \tilde{C}_i(t) \end{cases}$$

Note: if shape is kept constant, the standard point model is obtained.

Kinetic parameters:

⇒ reactivity

$$\rho(t) = \frac{\langle N_0^+ | \delta \hat{K} \varphi \rangle}{\langle N_0^+ | \hat{M} \varphi \rangle}$$

⇒ effective mean prompt-neutron generation time

$$\Lambda = \frac{\langle N_0^+ | \varphi \rangle}{\langle N_0^+ | \hat{M} \varphi \rangle}$$

⇒ effective delayed neutron fractions

$$\tilde{\beta}_i = \frac{\langle N_0^+ | \hat{M}_i \varphi \rangle}{\langle N_0^+ | \hat{M} \varphi \rangle}$$

Effective delayed neutron precursor concentrations

$$\tilde{C}_i(t) = \frac{\langle N_0^+ | \frac{\lambda_i}{4\pi} C_i \rangle}{\langle N_0^+ | \varphi \rangle} = \frac{1}{\Lambda} \frac{\langle N_0^+ | \frac{\lambda_i}{4\pi} C_i \rangle}{\langle N_0^+ | \hat{M} \varphi \rangle}$$

## Effective external source

$$\tilde{S}(t) = \frac{\langle N_0^+ | S \rangle}{\langle N_0^+ | \varphi \rangle} = \frac{1}{\Lambda} \frac{\langle N_0^+ | S \rangle}{\langle N_0^+ | \hat{M}\varphi \rangle}$$

For the numerical solution of the problem two time intervals are introduced:

- $\Rightarrow$  Shape interval (slow phenomena)  $\Delta t_\varphi$
- $\Rightarrow$  Amplitude interval (fast phenomena)

$\Delta t_P$

Discrete equations for the the shape:

$$\begin{aligned}
 T &= t_0 + \Delta t_\varphi \\
 P(T) \frac{\varphi(T) - \varphi(t_0)}{\Delta t_\varphi} + \varphi(T) \dot{P}(T) &= P(T) \hat{B} \varphi(T) + \\
 &\sum_{i=1}^6 \lambda_i \left[ \frac{\chi_i(E)}{4\pi} C_i(t_0) e^{-\lambda_i \Delta t_\varphi} + \right. \\
 &\left. \int_{t_0}^T P(t') \hat{M}_i \varphi(t_0) e^{-\lambda_i (T-t')} dt' + S(T) \right] \\
 \implies \varphi(T)
 \end{aligned}$$

In general,  $\varphi(T)$  will not satisfy the required normalization condition, namely:

$$\gamma(T) = \langle N_0^+ | \varphi(T) \rangle \neq \gamma(t_0) = \langle N_0^+ | \varphi(t_0) \rangle$$

An iterative process is necessary.

$$\begin{aligned}
& P(T) \frac{\varphi^{(n)}(T) - \varphi(t_0)}{\Delta t_\varphi} + \varphi^{(n)}(T) \dot{P}^{(n)}(T) = \\
& P(T) \hat{B} \varphi^{(n)}(T) + \sum_{i=1}^6 \lambda_i \left[ \frac{\chi_i(E)}{4\pi} C_i(t_0) e^{-\lambda_i \Delta t_\varphi} \right. \\
& \left. + \int_{t_0}^T P(t') \hat{M}_i \varphi(t_0) e^{-\lambda_i (T-t')} dt' + S(T) \right]
\end{aligned}$$

$$\Rightarrow \varphi^{(n)}(T)$$

$$\gamma^{(n)}(T) = \langle N_0^+ | \varphi^{(n)}(T) \rangle$$

$$\varphi^{(n+1/2)}(T) = \frac{\gamma(t_0)}{\gamma^{(n)}(T)} \varphi^{(n)}(T)$$

The derivative of the amplitude function is allowed to be discontinuous; it is updated according to:

$$\dot{P}^{(n+1)}(T) = P(T) \frac{\langle N_0^+ | \hat{B} \varphi^{(n+1/2)} \rangle}{\langle N_0^+ | \varphi^{(n+1/2)}(T) \rangle} + \sum_{i=1}^6 \lambda_i \frac{\langle N_0^+ | \frac{\chi_i}{4\pi} C_i \rangle}{\langle N_0^+ | \varphi^{(n+1/2)}(T) \rangle} + \frac{\langle N_0^+ | S(T) \rangle}{\langle N_0^+ | \varphi^{(n+1/2)}(T) \rangle}$$

## Source-Driven Problems:

Quasi-static needs to be adapted

The reference reactor is driven by an external source  $\Rightarrow$  the initial shape is assumed as solution of the steady-state equation:

$$(\hat{L}_0 + \hat{M}_0)N_0(r, E, \Omega) + S_0 = 0,$$

$$N_0(r_S, E, \Omega_{in}) = 0$$

How to define the adjoint?

a) introduce a multiplication eigenvalue in the adjoint equation

$$(\hat{L}_0^+ + \frac{1}{k_0} \hat{M}_0^+) N_{0,cr}^+(r, E, \Omega) = 0,$$

$$N_{0,cr}^+(r_S, E, \Omega_{out}) = 0$$

b) introduce an adjoint source

$$(\hat{L}_0^+ + \hat{M}_0^+) N_{0,s}^+(r, E, \Omega) + S^+ = 0,$$

$$N_{0,s}^+(r_S, E, \Omega_{out}) = 0$$

Problem: how to define the adjoint source?

Possible solution:

⇒ if importance is defined as the number of fission neutrons to be produced per neutron injected at a point in phase space, adjoint source is

$$S^+ = \nu \Sigma_f$$

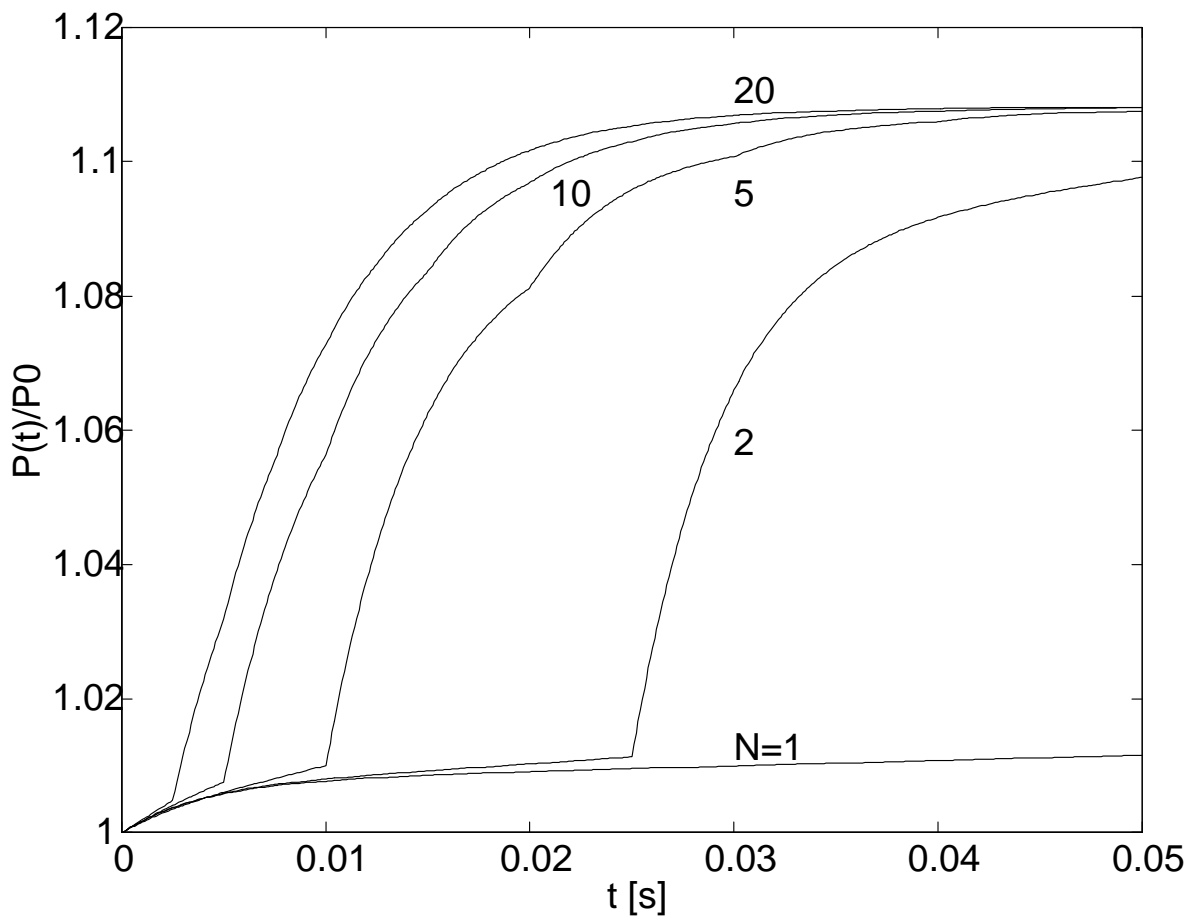
Consequence on the reactivity of the system:

$$\rho = \rho_0 + \rho_p$$

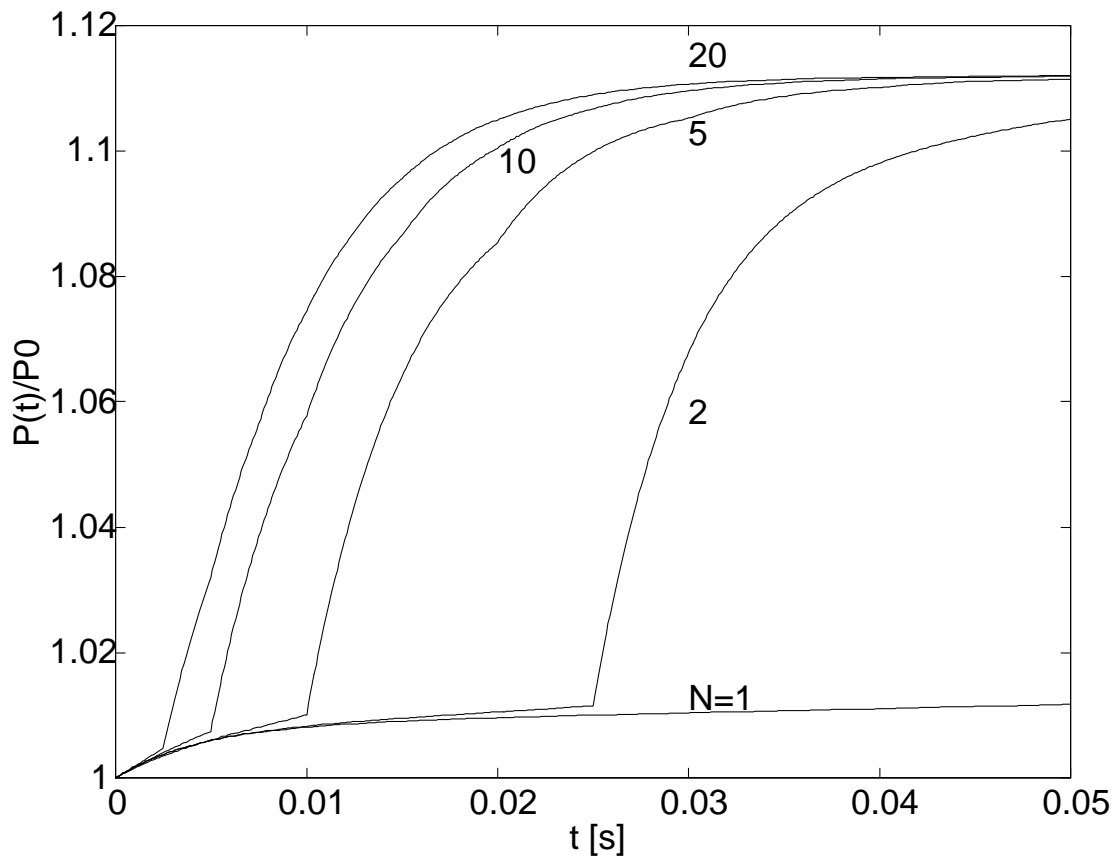
$\rho_p$  is the perturbation reactivity

$\rho_0$  is connected to the initial subcriticality level

$$\rho_0 = \frac{\langle N_{0,cr}^+ | \hat{M}_0 \varphi \rangle}{\langle N_{0,cr}^+ | \hat{M} \varphi \rangle} \left( \frac{k-1}{k} \right)$$
$$\rho_0 = - \frac{\langle S^+ | \varphi \rangle}{\langle N_{0,s}^+ | \hat{M} \varphi \rangle}$$



Power: weight function as source  
adjoint.



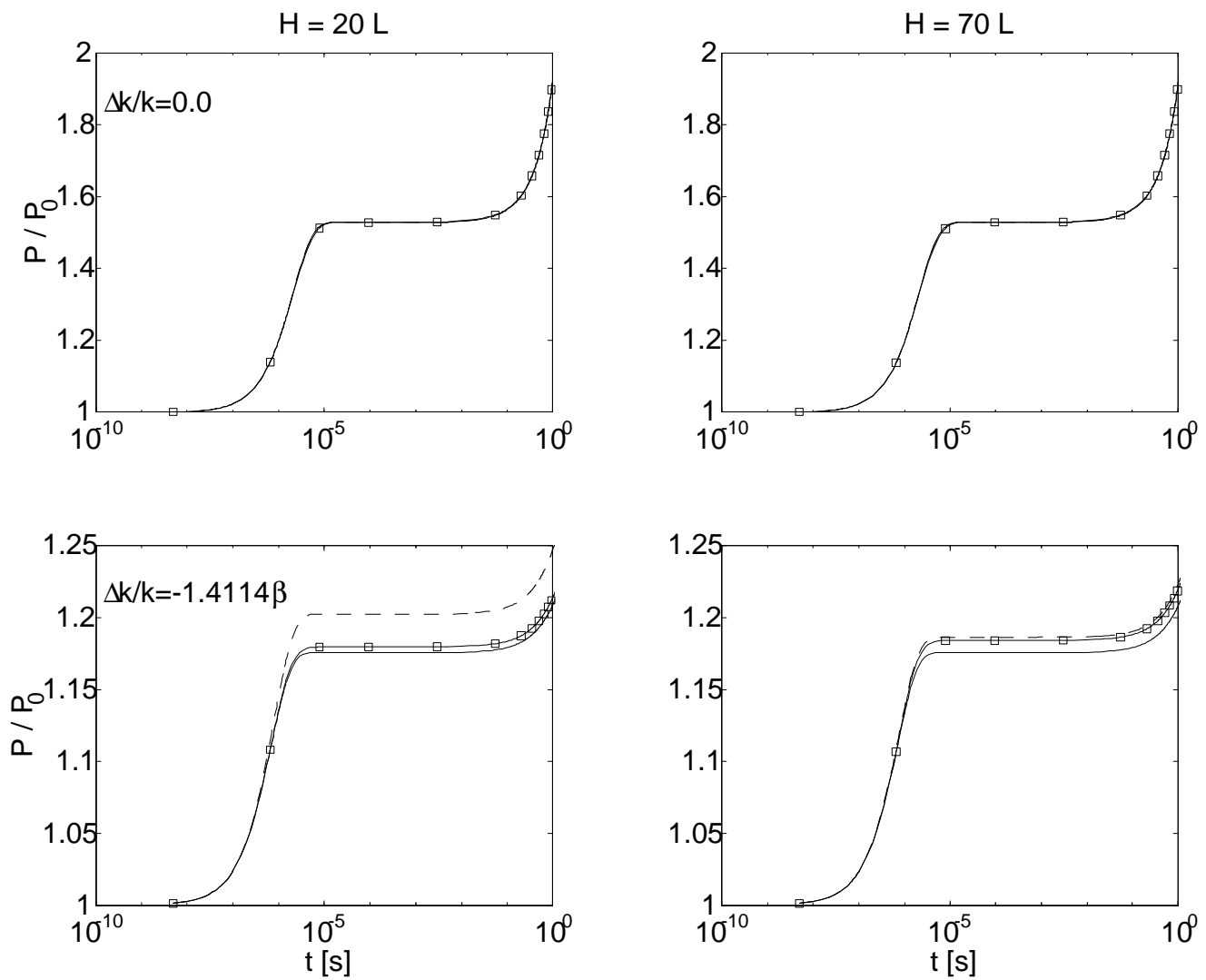
Power: weight function as critical adjoint.

# DISCUSSION ON THE CHOICE OF THE WEIGHTING FUNCTION IN SEPARATION SCHEMES

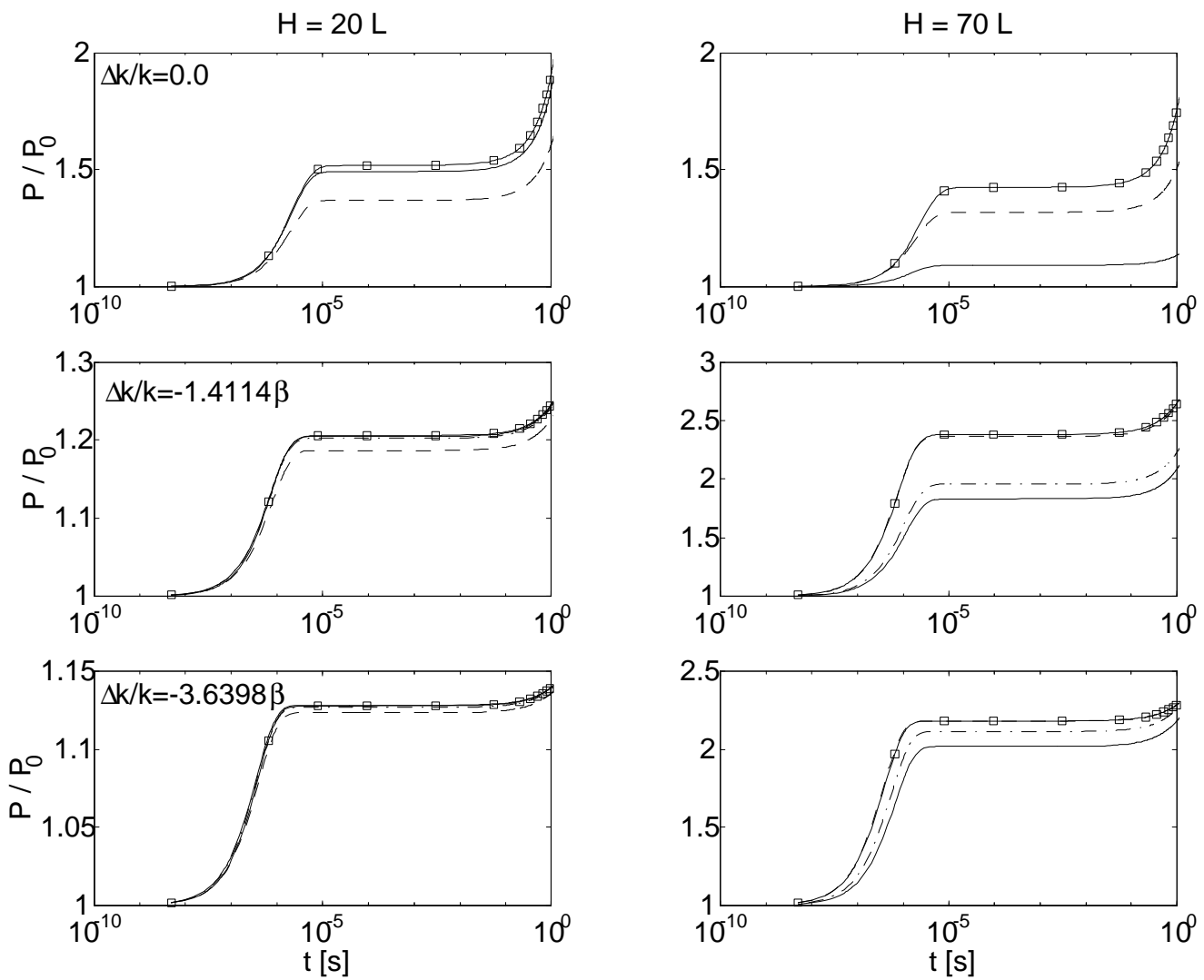
It is an open problem for subcritical systems:

⇒ Evidence problem, rather than propose solutions;

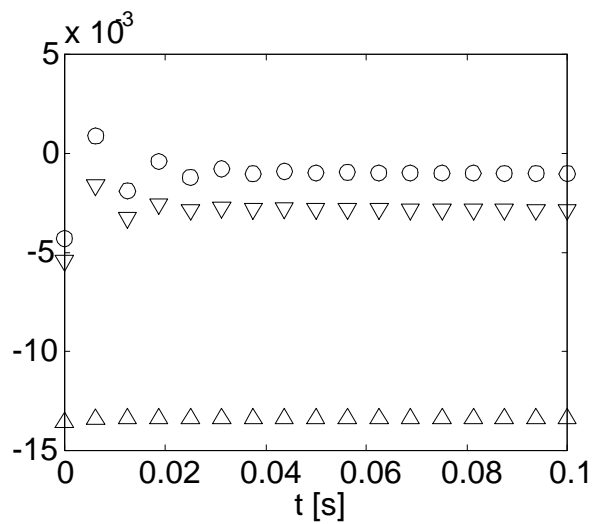
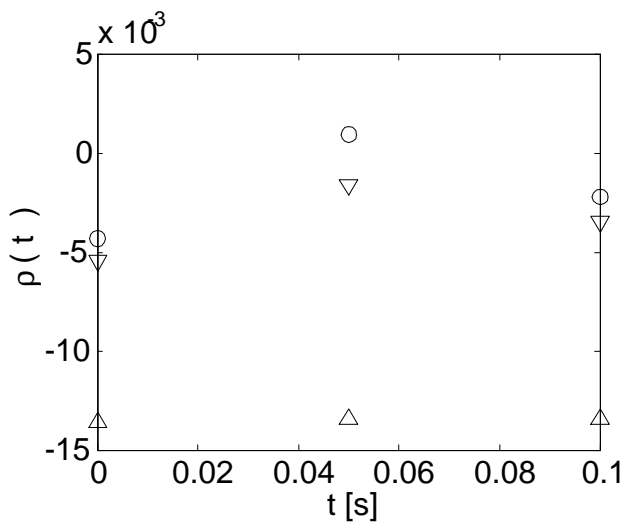
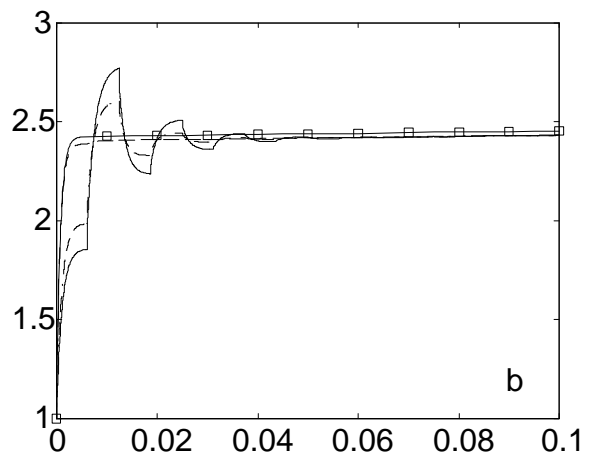
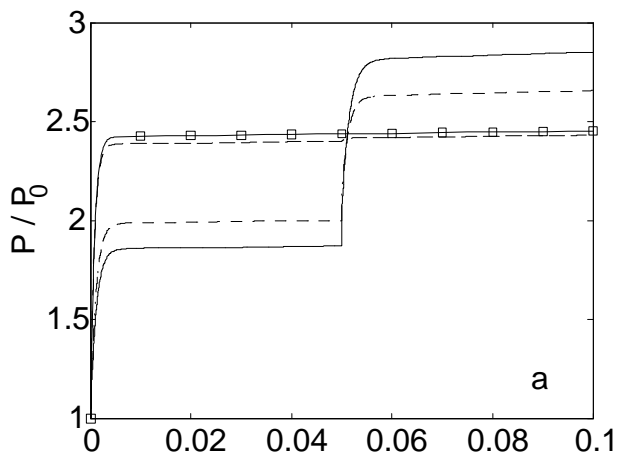
⇒ Reasonable possibilities: critical adjoint or the adjoint driven by the fission cross-section as a source.



Homogeneous perturbation, systems with different  $k$ . Top: initial critical system; bottom: initial subcritical system,  $k = 0.98$ . Squares: reference; broken line: constant adjoint; solid line: critical adjoint; dot-point line: source-driven adjoint.



Comparison of the power evolutions calculated using different weighting functions in response to a localized perturbation. Squares: reference results; broken line: constant adjoint; solid line: critical adjoint; dot-point line: source-driven adjoint.



Quasi-static calculations with  $H = 70L$  and  $\Delta k/k = -1.4114\beta$ , compared to the full spatial results. One shape update for graphs a), 15 recalculations for case b). Reactivity at the bottom;  $\nabla$ : source adjoint;  $\triangle$ : constant;  $\circ$ : critical adjoint.

# COMPUTATIONAL TOOL

The neutronic module solves multi-group diffusion equations in cylindrical geometry

The thermal calculation is performed by the channel code TIESTE developed at ENEA (Italy) for accelerator-driven systems cooled by lead-bismuth

The instants for the thermal calculation are chosen according to the power change of the system

The average values of the power are computed for pre-defined zones of the multiplying structure and input into the thermal module

The thermal code can represent a single reactor channel with fuel, clad and coolant

At the end of the thermal calculation, the cross-sections are updated according to a linear interpolation process between tables of data generated at different fuel and coolant temperatures

# DEVELOPMENTS

The quasi-static method proves to be a powerful and computationally efficient tool for the analysis of source-driven systems

Accurate predictions of the transient behavior in non-linear conditions dominated by thermal feed-back can be attained

Development:

⇒ An improvement: the **multipoint** method

Balance equations in discretized form:

$$\left\{ \begin{array}{l} \frac{1}{v_m} \frac{d\phi_{nm}}{dt} = \sum_{n'} \sum_{m'} k_{nm,n'm'} \phi_{n'm'} + \\ \sum_{i=1}^6 \lambda_i \chi_{i,m} C_{i,n} + S_{nm} \\ \frac{dC_{i,n}}{dt} = \beta_i \sum_{m'} f_{nm'} \phi_{nm'} - \lambda_i C_{i,n} \quad i = 1, 2, \dots, 6 \end{array} \right.$$

$$\phi_{nm}(t) = \phi(\mathbf{r}_n, V_m, t) \quad C_{i,n}(t) = C_i(\mathbf{r}_n, t)$$

$$\phi_{nm}(t) = A_{NM}(t) \varphi_{nm}(t) \quad \mathbf{r}_n, V_m \in \Gamma_{NM}$$

definition of a regionwise inner product

$$\langle w | g \rangle = \left[ \sum_n \sum_m \right]_{NM} w_{nm} g_{nm}$$

Introduce factorization into the balance equations:

$$\left\{ \begin{array}{l} \frac{1}{v_m} \varphi_{nm} \frac{dA_{NM}}{dt} + \frac{1}{v_m} A_{NM} \frac{d\varphi_{nm}}{dt} = \\ \sum_{N'} \sum_{M'} \left[ \sum_{n'} \sum_{m'} \right]_{N'M'} k_{nm,n'm'} \varphi_{n'm'} A_{N'M'} + \\ \sum_{i=1}^6 \lambda_i \chi_{i,m} C_{i,n} + S_{nm} \\ \\ \frac{dC_{i,n}}{dt} = \beta_i \sum_{M'} \left[ \sum_{m'} \right]_{M'} f_{nm'} \varphi_{nm'} A_{NM'} - \lambda_i C_{i,n} \\ i = 1, 2, \dots, 6, \quad \mathbf{r}_n, V_m \in \Gamma_{NM} \end{array} \right.$$

Projection: multiply by  $w_{nm}$  and sum on  $NM$

Normalization condition

$$\frac{d}{dt} \left[ \sum_n \sum_m \right]_{NM} w_{nm} \frac{1}{v_m} \varphi_{nm}(t) = \frac{d}{dt} \gamma_{NM} = 0$$

## Point-to-point transfer term

$$\left[ \sum_n \sum_m \right]_{NM} w_{nm} \sum_{N'} \sum_{M'} \left[ \sum_{n'} \sum_{m'} \right]_{N'M'} \dots$$

$$k_{nm,n'm'} \varphi_{n'm'} A_{N'M'} =$$

$$\sum_{N'} \sum_{M'} \left[ \sum_n \sum_m \right]_{NM} \dots$$

$$\left( w_{nm} \left[ \sum_{n'} \sum_{m'} \right]_{N'M'} k_{nm,n'm'} \varphi_{n'm'} \right) A_{N'M'} =$$

$$\sum_{N'} \sum_{M'} K_{NM,N'M'}^* A_{N'M'}$$

Multipoint equations:

$$\left\{ \begin{array}{l} \frac{dA_{NM}}{dt} = \sum_{N'} \sum_{M'} K_{NM, N'M'} A_{N'M'} + \\ \quad \sum_{i=1}^6 \lambda_i C_{i, NM} + S_{NM} \\ \frac{dC_{i, NM}}{dt} = \beta_i \sum_{M'} F_{i, NM, M'} A_{NM'} - \lambda_i C_{i, NM} \\ \quad i = 1, 2, \dots, 6, \end{array} \right.$$

point-to-point coupling coefficients

$$K_{NM,N'M'} = \frac{1}{\gamma_{NM}} \left[ \sum_n \sum_m \right]_{NM} \left( w_{nm} \left[ \sum_{n'} \sum_{m'} \right]_{N'M'} k_{nm,n'm'} \varphi_{n'm'} \right)$$

effective multipoint source

$$S_{NM} = \frac{1}{\gamma_{NM}} \left[ \sum_n \sum_m \right]_{NM} w_{nm} S_{nm}$$

effective multipoint delayed precursor concentrations

$$C_{i,NM} = \frac{1}{\gamma_{NM}} \left[ \sum_n \sum_m \right]_{NM} w_{nm} \chi_{i,m} C_{i,n}$$

delayed neutron production coefficients

$$F_{i,NM,M'} = \frac{1}{\gamma_{NM}} \left[ \sum_n \sum_m \right]_{NM} w_{nm} \chi_{i,m} \beta_i \left[ \sum_{m'} \right]_{M'} f_{nm'} \varphi_{nm'}$$

Multipoint can be included into a quasi-static scheme

Solution of the "slow" shape equation

$$T = t_0 + \Delta t_\varphi$$

$$\frac{1}{v_m} \dot{A}_{NM}(T) \varphi_{nm}(T) + \frac{1}{v_m} A_{NM}(T) \frac{\varphi_{nm}(T) - \varphi_{nm}(t_0)}{\Delta t_\varphi} =$$

$$\sum_{N'} \sum_{M'} \left[ \sum_{n'} \sum_{m'} \right]_{N'M'} k_{nm,n'm'}(T) \varphi_{n'm'}(T) A_{N'M'}(T) \\ + \sum_{i=1}^6 \lambda_i \chi_{i,m} C_{i,n}(T) + S_{nm}(T)$$

$$C_{i,n}(T) = C_{i,n}(t_0) e^{-\lambda_i \Delta t_\varphi} +$$

$$\int_{t_0}^T \beta_i \sum_{M'} \left[ \sum_{m'} \right]_{M'} f_{nm'}(T) \varphi_{nm'}(t_0) A_{N'M'}(t') \times$$

$$e^{-\lambda_i (T-t')} dt'$$

## Iteration scheme

$$\frac{1}{v_m} \dot{A}_{NM}^{(l)}(T) \varphi_{nm}^{(l)}(T) + \frac{1}{v_m} A_{NM}(T) \frac{\varphi_{nm}^{(l)}(T) - \varphi_{nm}(t_0)}{\Delta t_\varphi} =$$

$$\sum_{N'} \sum_{M'} \left[ \sum_{n'} \sum_{m'} \right]_{N'M'} k_{nm, n'm'}(T) \varphi_{n'm'}^{(l)}(T) A_{N'M'}(T) +$$

$$\sum_{i=1}^6 \lambda_i \chi_{i,m} C_{i,n}(T) + S_{nm}(T)$$

$$\gamma_{NM}^{(l)}(T) = \left[ \sum_n \sum_m \right]_{NM} w_{nm} \frac{1}{v_m} \varphi_{nm}^{(l)}(T)$$

$$\varphi_{nm}^{(l+1/2)}(T) = \frac{\varphi_{nm}^{(l)}(T)}{\gamma_{NM}^{(l)}(T)} \gamma_{NM}(t_0)$$

$$\dot{A}_{NM}^{(l+1)}(T) = \gamma_{NM}^{(l+1/2)}(T) \sum_{N'} \sum_{M'} \left[ \sum_n \sum_m \right]_{NM} \dots$$

$$\left( w_{nm} \left[ \sum_{n'} \sum_{m'} \right]_{N'M'} k_{nm,n'm'}(T) \varphi_{n'm'}^{(l+1/2)}(T) \right) A_{N'M'}$$

$$+ \frac{\left[ \sum_n \sum_m \right]_{NM} w_{nm} \chi_{i,m} C_{i,n}(T)}{\gamma_{NM}^{(l+1/2)}(T)} +$$

$$\frac{\left[ \sum_n \sum_m \right]_{NM} w_{nm} S_{nm}(T)}{\gamma_{NM}^{(l+1/2)}(T)}$$

## The Adiabatic Scheme

Use as equation for the determination of the shape a static equation with an eigenvalue (homogeneous problem):

$$(\hat{L}(t) + \frac{1}{k(T)}\hat{M}(T))N_k(r, E, \Omega; T) = 0,$$

$$N_k(r_S, E, \Omega_{in}) = 0$$

normalization condition:

$$\langle N_0^+ | N_k(T) \rangle = \langle N_0^+ | N_0 \rangle$$

reactivity takes the usual static form:

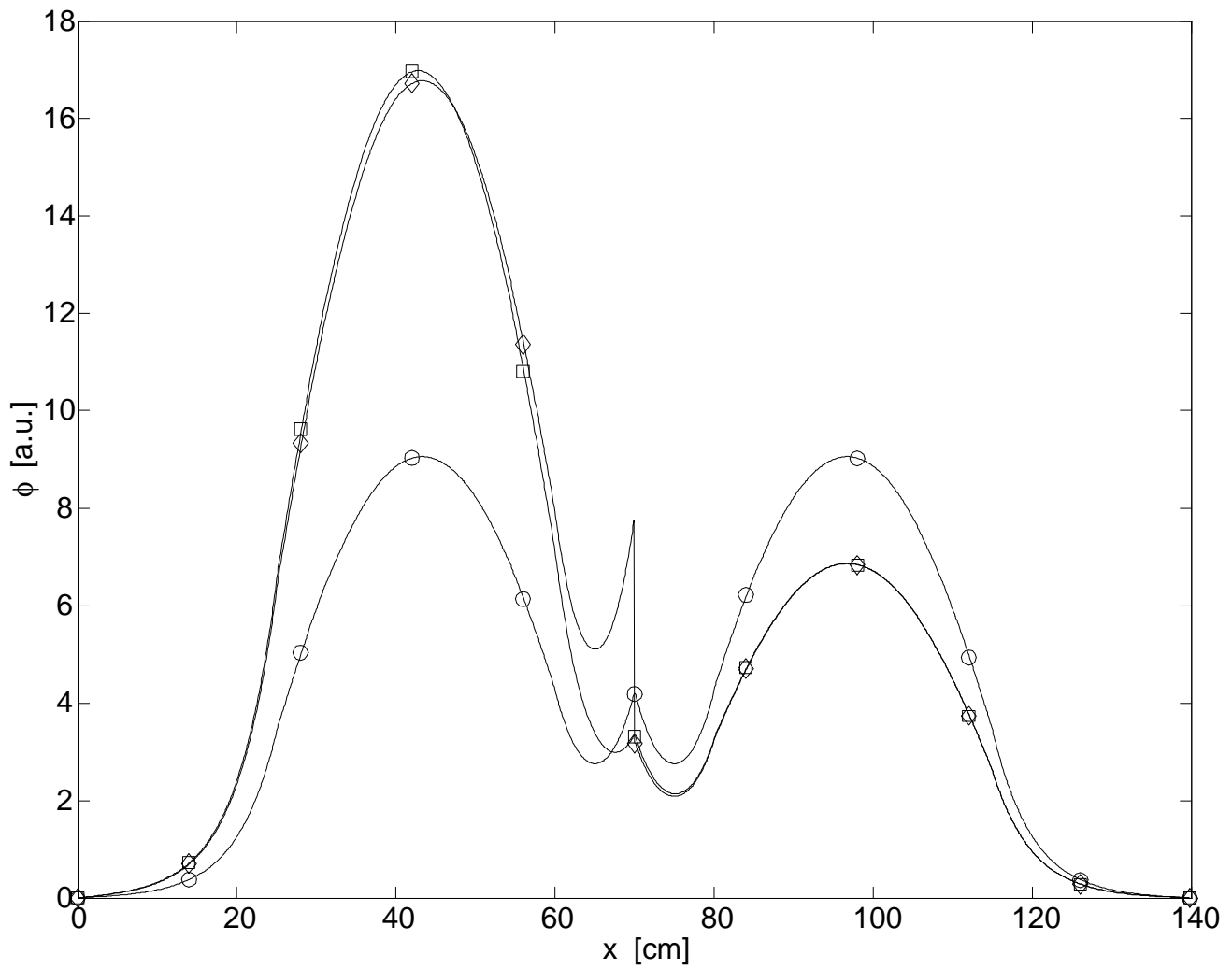
$$\rho(T) = \frac{k(T) - 1}{k(T)}$$

## **Concluding remarks**

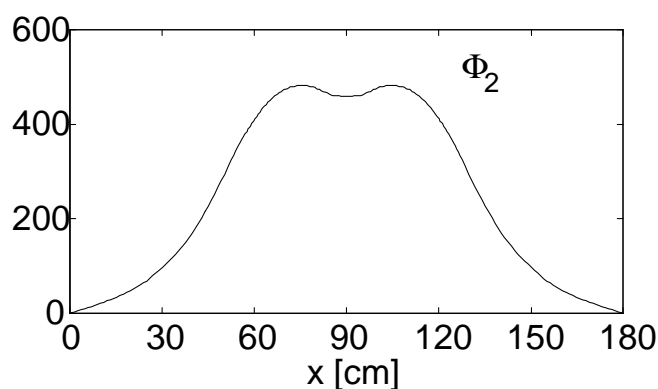
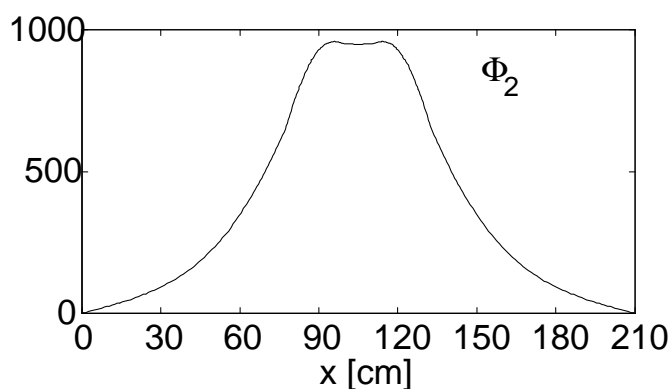
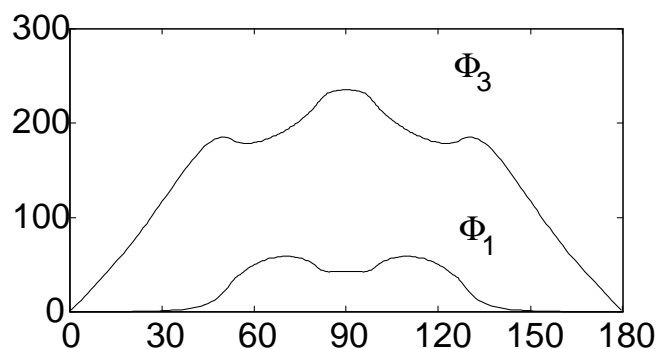
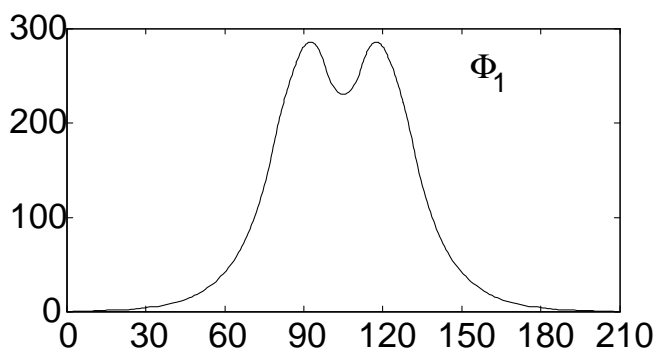
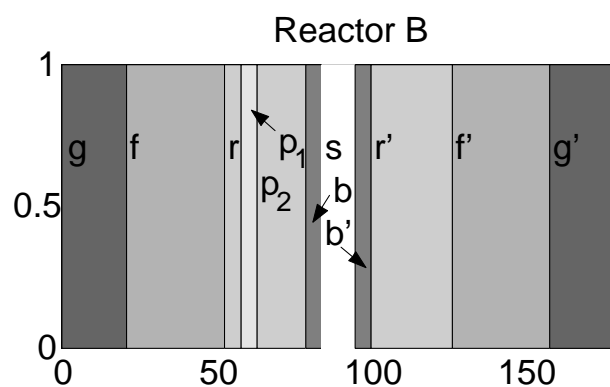
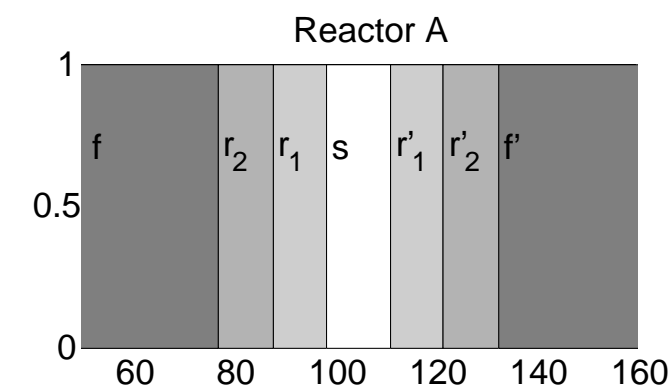
- multipoint is effective in many reactor kinetics problems

- the method can easily be included within quasi-statics, greatly enhancing its performance

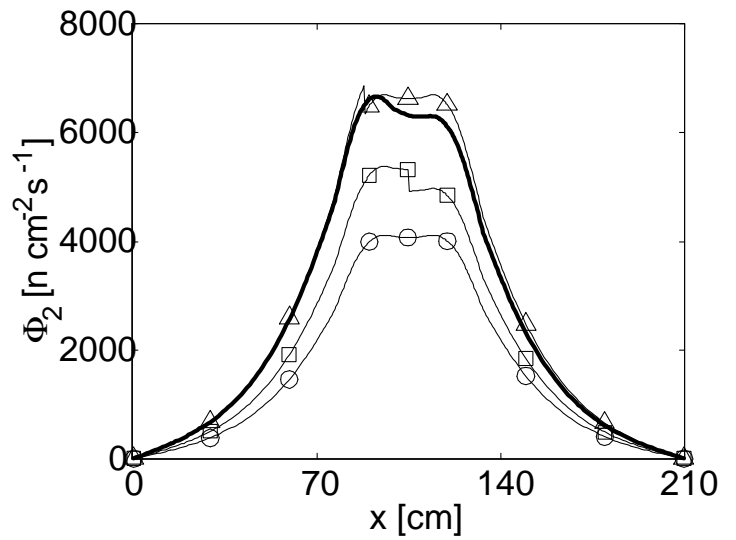
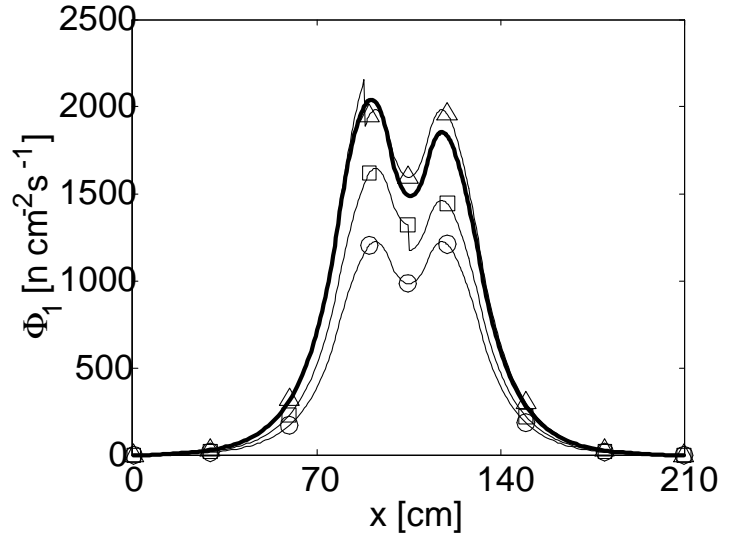
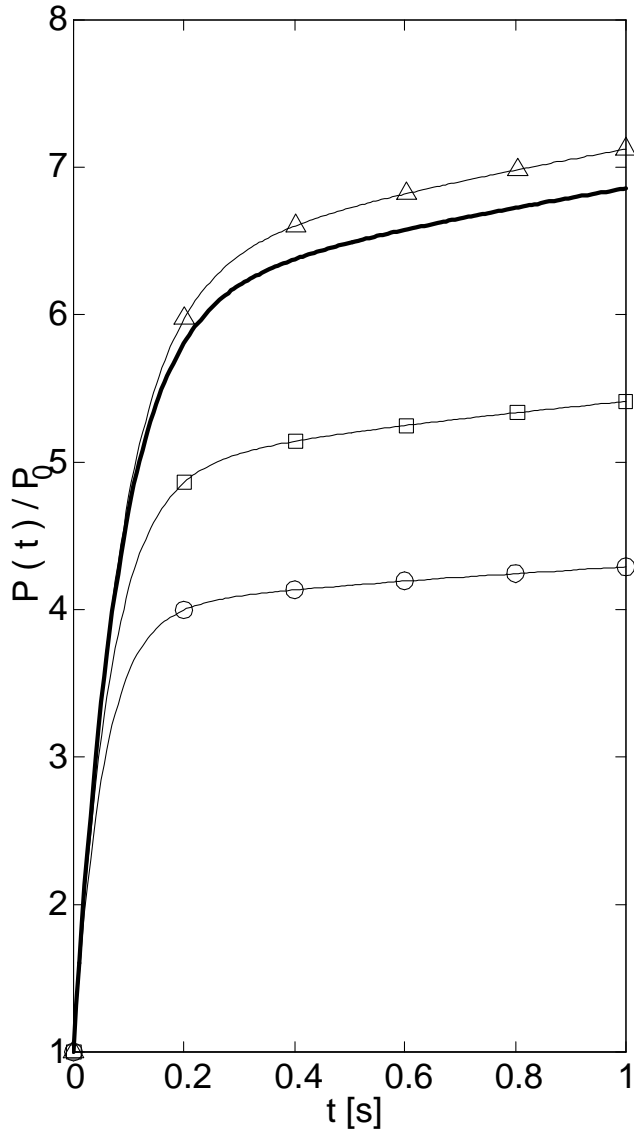
- development: apply multipoint to angular schemes in transport calculations



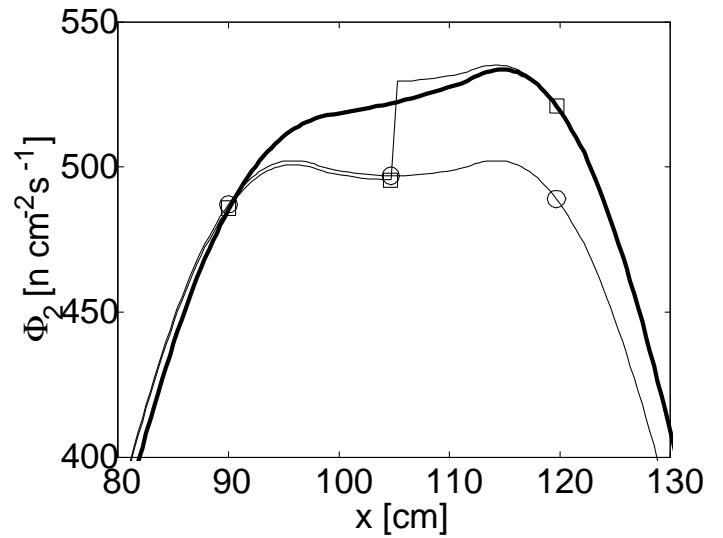
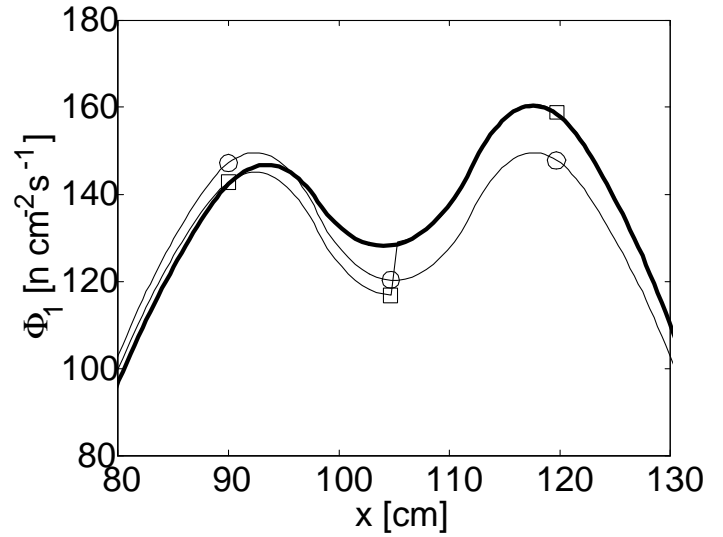
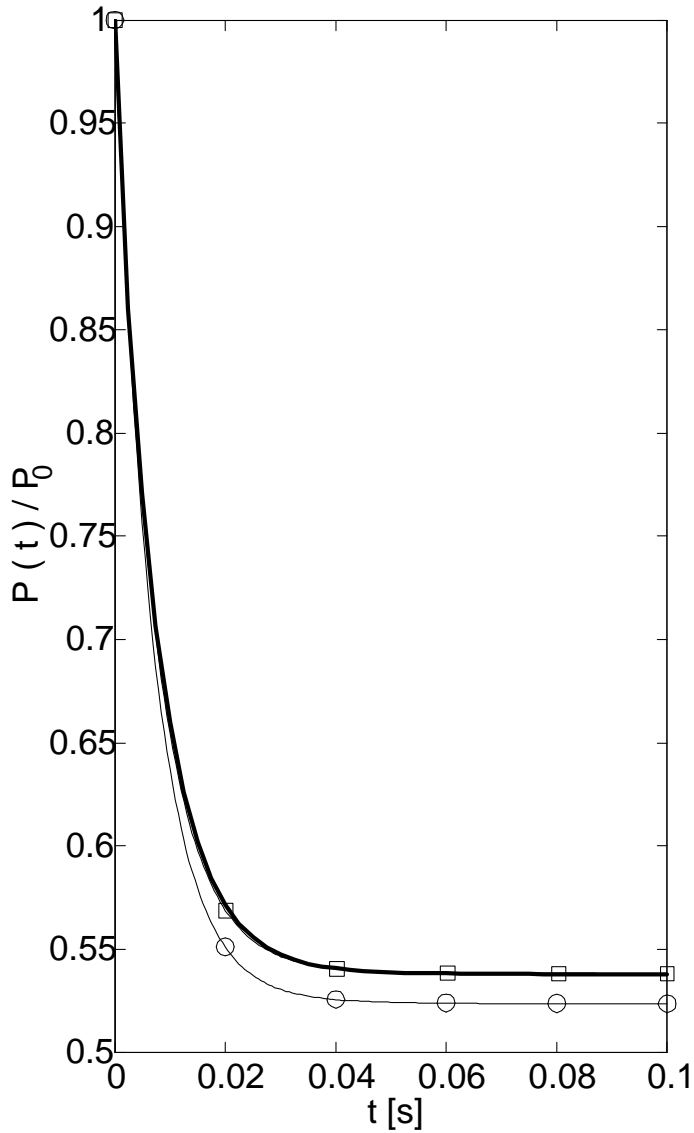
1. Flux at end of transient:  $\circ$  : point kinetics;  $\diamond$  : 2-point kinetics;  $\square$  : exact solution.



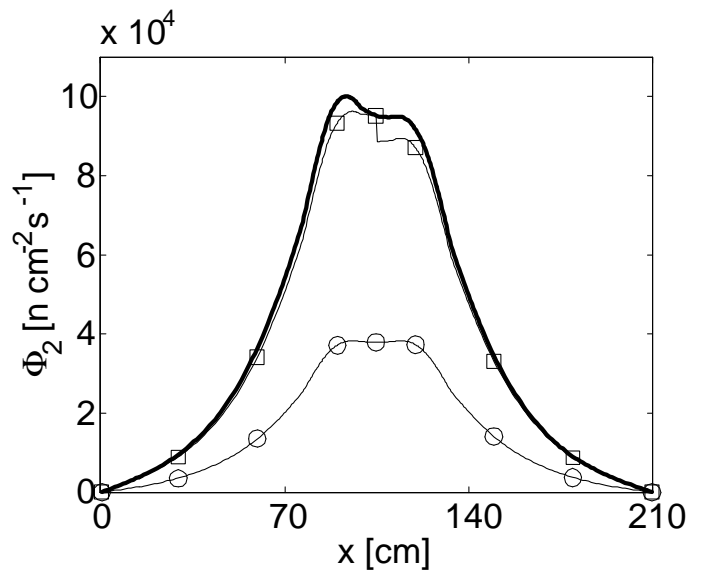
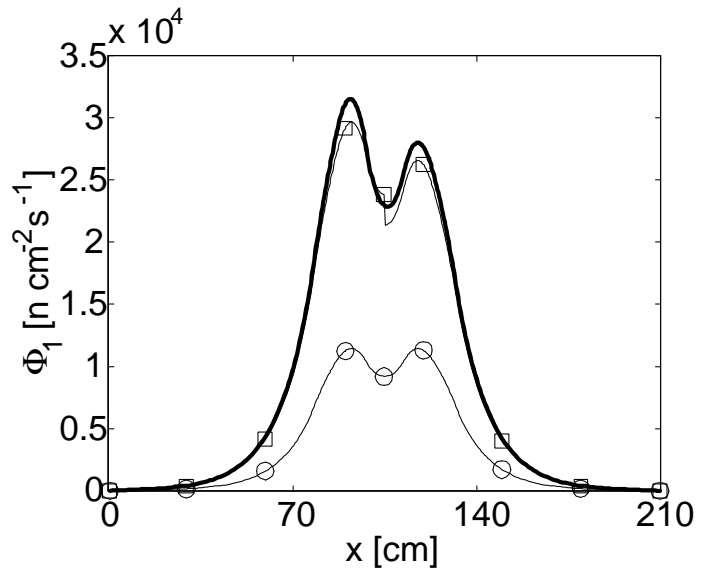
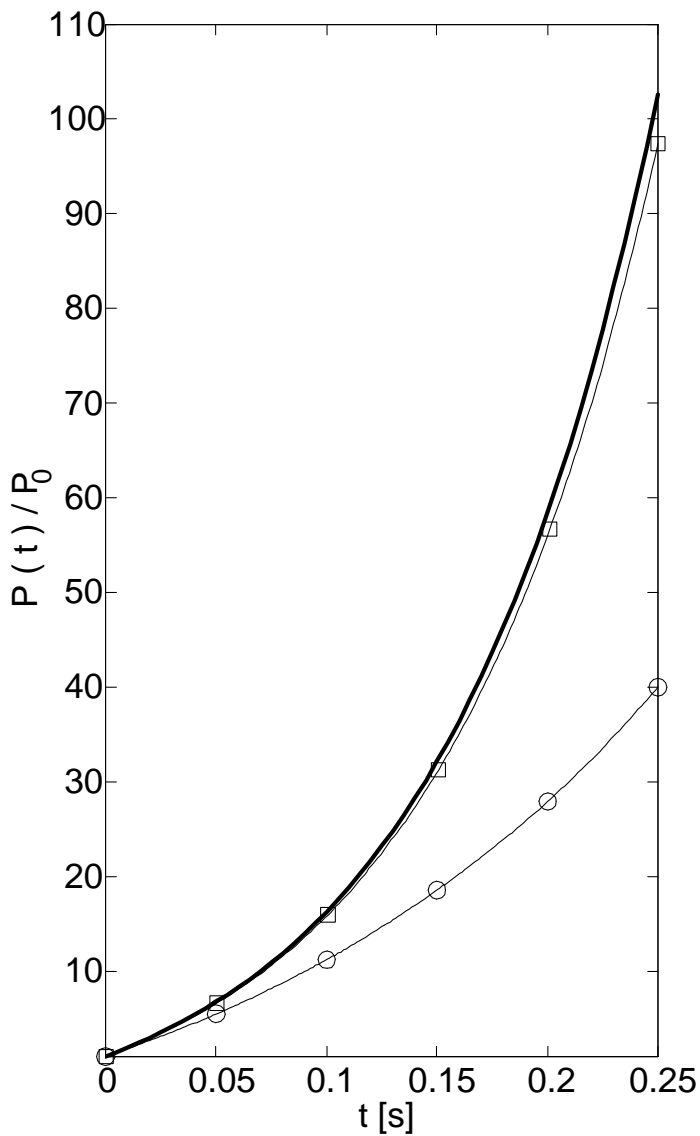
2. Material configurations. s: source channel; r, r': multiplying zones, Masurca type; r<sub>1</sub>, r'<sub>1</sub>: multiplying zones, Myrrha type (low enrichment); r<sub>2</sub>, r'<sub>2</sub>: multiplying zone, Myrrha type (high enrichment); f, f': reflectors; b, b': lead buffers; p<sub>1</sub>, p<sub>2</sub>: perturbed zones; g, g': shields. Initially, r, p<sub>1</sub> and p<sub>2</sub> have the same properties as r'. Flux in steady state, 2-group diffusion for A and 3-group diffusion for B.



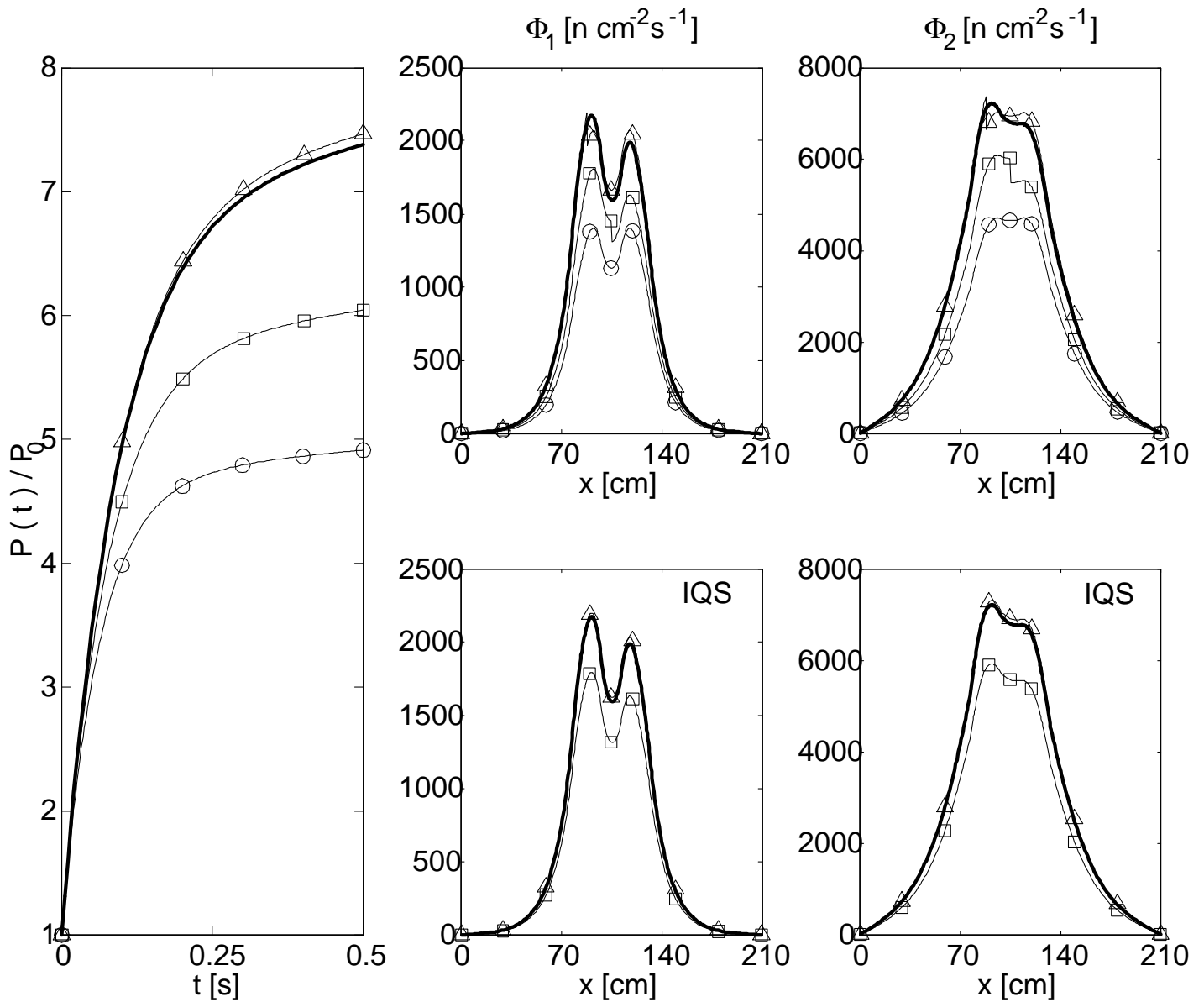
3. Transient caused by a  $\delta\Sigma_{a1} = -0.1\delta\Sigma_1$  in the  $r_2$  zone of system A. The multiplication constant becomes  $k_{eff} = 0.998447$ . On the right, the fluxes at the last time considered in the transient ( $t = 1$  s) are plotted.



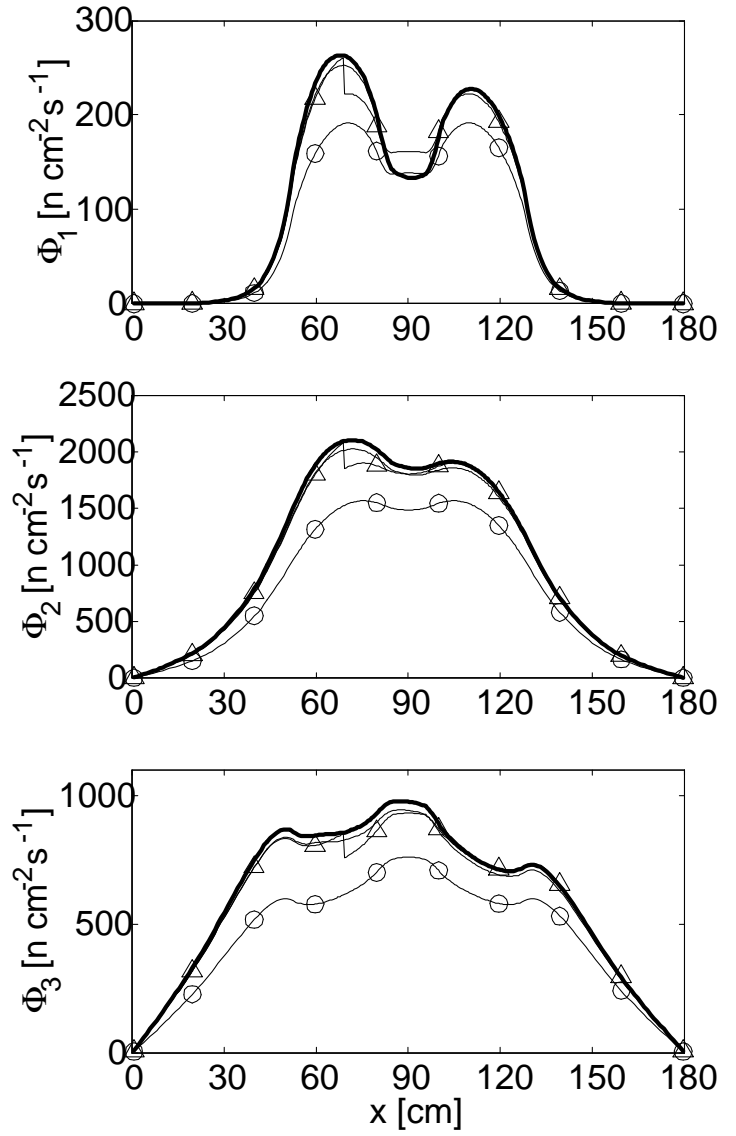
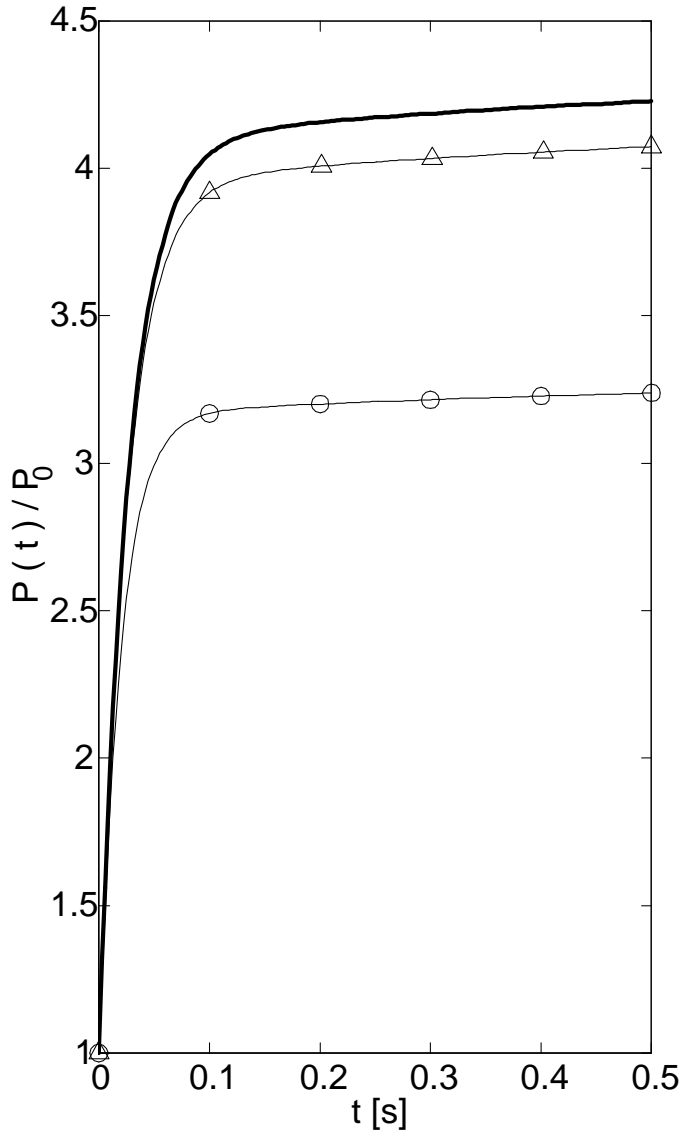
4. Transient caused by a  $\delta\Sigma_{a1} = +0.05\delta\Sigma_1$  in the  $r_1$  and  $r_2$  zones of system A. The multiplication constant becomes  $k_{eff} = 0.961767$ . Final fluxes on the right.



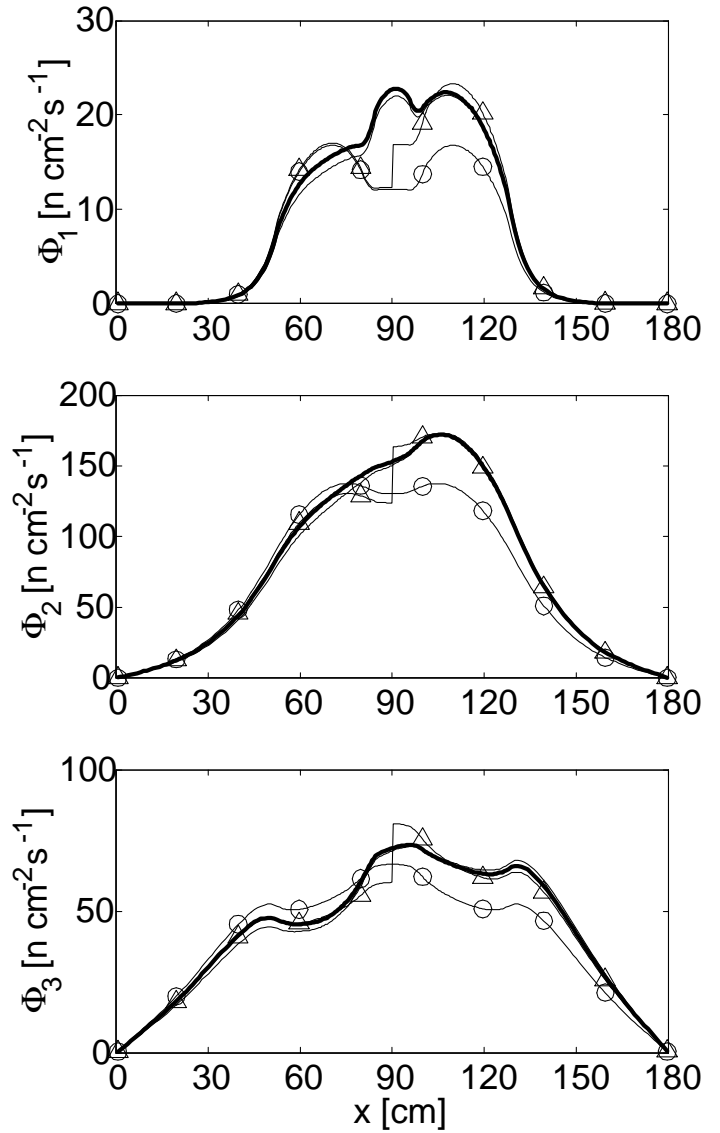
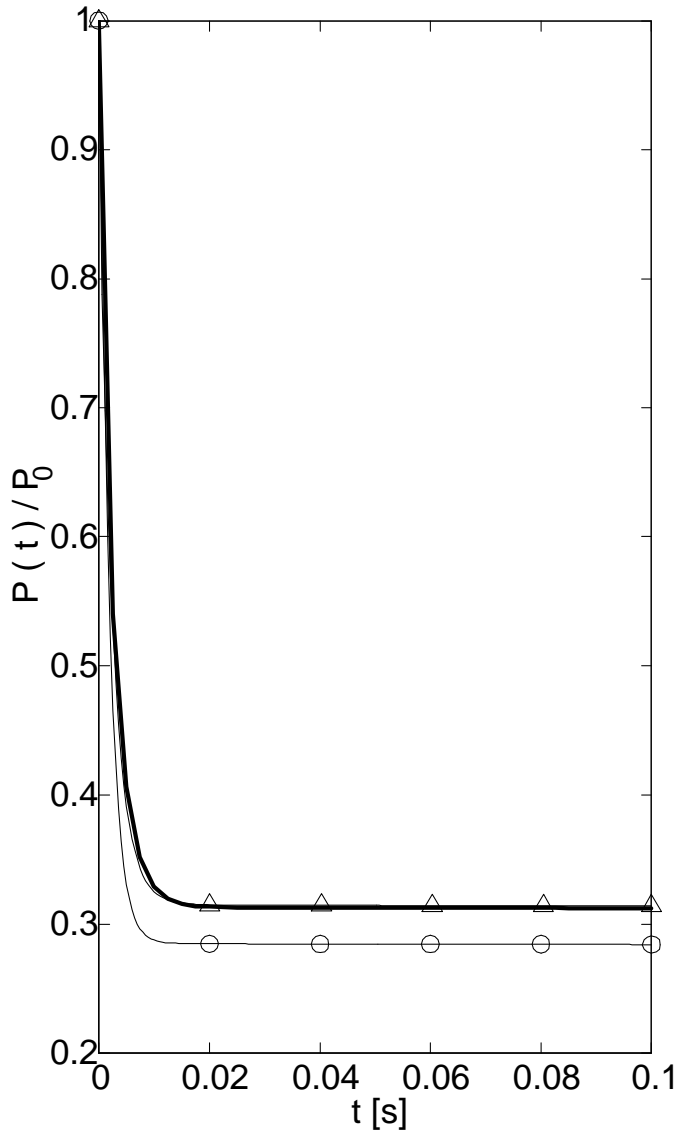
5. Transient caused by a  $\delta\Sigma_{a1} = +0.1\delta\Sigma_1$  in the  $r_1$  zone of system A. The multiplication constant becomes  $k_{eff} = 1.00489$ .



6. Transient caused by a  $\delta\Sigma_{a1} = -0.05\delta\Sigma_1$  and  $\delta\Sigma_{a2} = -0.1\delta\Sigma_2$  in the  $r_2$  zone of system A. The multiplication constant becomes  $k_{eff} = 0.998850$ . In the right on the top, comparison of reference, point and two-point kinetics can be seen. At the bottom, IQS results are reported.



7. Transient caused by a  $\delta\Sigma_{ag} = -0.2\delta\Sigma_g$  for all groups in the  $p_1$  zone of system B. The multiplication constant becomes  $k_{eff} = 0.994264$ .



8. Transient caused by a  $\delta\Sigma_{ag} = +0.2\delta\Sigma_g$  for all groups in the  $p_2$  zone of system B. The final multiplication constant is  $k_{eff} = 0.912024$ .