Nonlinear Ion-Acoustic Waves in a Plasma Consisting of Warm Ions and Isothermal Distributed Electrons.

A. M. Abourabiaa, K. M. Hassan, Rabab A. Shaheinb.

Department of Mathematics, Faculty of Science, Menoufiya University, Shebin El-Kom 32511, Egypt.

E-mail: aam_abourabia@yahoo.com
brababshahein@yahoo.com

ABSTRACT

The formation of (1+1) dimensional ion-acoustic waves (IAWs) in an unmagnetized collisionless plasma consisting of warm ions and isothermal distributed electrons is investigated. The electrodynamics system of equations are solved analytically in terms of a new variable $\xi = kx - \omega t$, where $k = k(\omega)$ is a complex function, at a fixed position. The analytical calculations gives that the critical value $\frac{T}{T_d} \approx 0.25$ distinguishes between the linear and nonlinear characters of IAW within the nanosecond time scale. The flow velocity, pressure, number density, electric potential, electric field, mobility and the total energy in the system are estimated and illustrated.

Keywords: Collisionless Plasma/ Ion-Acoustic Waves/ Nonlinear Phenomena.

INTRODUCTION

During the last thirty eight years, there has been an important contribution in the field of nonlinear plasma physics. Several effects such as ion temperature, ion density gradient, presence of third species (dust), oblique propagation, etc., have been investigated on the structure of ion acoustic waves (IAWs). Nonlinear ion-sound waves have been studied by a number of authors. The transverse stability of strongly nonlinear ion-acoustic structures, described by the general Sagdeev potential, is investigated. It is shown that the multi-scale stability analysis by (1) for solitary waves, described by a weakly dispersive and nonlinear equations, can be generalized for strongly nonlinear systems. The dispersion relations for transverse ion oscillations have been obtained in (2). Xue Jukui using the reductive perturbation method (3) derived a nonlinear Schrödinger equation governing the slow modulation of IAW in an unmagnetized plasma consisting of warm adiabatic ions and non-thermal electrons. He found that a plasma with non-thermally distributed electrons would significantly changes the modulational instability domain in the $k-\sigma$ plane in one dimensional Cartesian coordinate. Berger et al.(4), studied the frequency and the damping of slow and fast ion acoustic waves in collisional and collisionless two-species plasma as an example of a mixture of hydrogen and xenon. They proved that the damping rate of ion acoustic wave is very sensitive to the increasing of ion temperature $\sigma$. Linear and nonlinear propagation of ion-acoustic waves are theoretically investigated in a multicomponent plasma consisting of electrons, positive and negative ions bounded in a cylindrical waveguide by Kalyan et al.(5). The effect of nonlinearity on the ion-acoustic wave is investigated through the derivation of the effective potential (Sagdeev potential) and the results are discussed graphically. They discussed the relation between the phase velocity and concentration of negative ions. Esfandyari-Kalejahi et al.(6), studied the amplitude modulation of ion-acoustic waves (IAWs) in a plasma...
consisting of warm ions and a cold electron beam both theoretically and numerically, where the
perturbations parallel to the carrier IAW propagation direction have been investigated together
with the stability analysis, based on a nonlinear Schrödinger equation (NLSE).

In this paper we study the nonlinear ion acoustic waves occur in plasma when the electrons
can screen the fluctuations of the ion charge density by the Debye shielding, so that the pressure
gradient drives the oscillation. The waves are acoustic under the condition \( k \lambda_{De} \ll 1 \). We will
derive an analytical solution of the nonlinear electrodynamics equations of our system by using
the separation of variables.

Nomenclature:
\( \gamma \) = The ratio of the specific heat at constant pressure to the specific heat at constant volume
(for the one dimensional model \( \gamma = 3 \)),
\( \varepsilon \) = The total energy of IAW,
\( \lambda_{De} \) = The effective Debye length, (Here \( \lambda_{De} \) is the electrostatic shielding distance of the electrons,
the electron Debye length)
\( \sigma \) = The ratio of the ions temperature to the electrons temperature \( (T_i/T_{eff}) \),
\( \Phi, \phi \) = The electrostatic potential,
\( \omega \) = The frequency of IAW,
\( \omega_{ni} \) = The ion-plasma frequency,
\( C_S \) = The speed of sound,
\( E \) = The electric field,
\( k \) = The wave number of IAW,
\( m_i \) = The ion mass,
\( n_i \) = The ion density,
\( n_0 \) = The common equilibrium plasma density,
\( P_i \) = The ion pressure,
\( P_0 \) = The ion equilibrium pressure,
\( T_i \) = The ion equilibrium temperature,
\( T_{eff} \) = The effective temperature of electrons,
\( u \) = The fluid velocity (the mean velocity of ions in a plasma),
\( V_{ph} \) = The phase velocity of IAW,
\( V_g \) = The group velocity of IAW.

THE PHYSICAL SITUATION AND MATHEMATICAL FORMULATION.

Below we consider a fully ionized and unmagnetized collisionless plasma. The collisionless
condition is satisfied when \( \nu_c / \omega_p \ll 1 \) (where \( \nu_c \) is the collision frequency and \( \omega_p \) is the ion-
plasma frequency), where in the collisionless limit, the coupling between species occurs only in
the self-consistent electric field. To ignore the Landau damping effects , it is assumed that \( T_i \ll T_{eff} \). The nonlinear evolution of the IAWs in multicomponent plasma consisting of warm ions
and non-thermal electrons is described by the electrodynamics set of equations. The wave in this
plasma is assumed to propagate along the x-direction.

We use the ion fluid equations for a one-dimensional collisionless unmagnetized plasma. The adopted electrodynamics model of the ion-plasma comprises the essential equations of continuity, momentum, energy equations of ions, and supplemented by Maxwell’s equation.

The nonlinear equations of the ion fluid in non-dimensional form are:

The ion continuity equation \(^{(2,3,9)}\)
\[
\frac{\partial n_i}{\partial t} + \frac{\partial (n_i u_i)}{\partial x} = 0
\]  

The ion momentum equation \(^{(3,10)}\)
\[
\frac{\partial n_i}{\partial t} + n_i u_i \frac{\partial u_i}{\partial x} = -n_i \frac{\partial \varphi}{\partial x} - \sigma \frac{\partial P_i}{\partial x}, \quad (2)
\]

The ion energy equation \((3, 11)\)

\[
\frac{\partial P_i}{\partial t} + u_i \frac{\partial P_i}{\partial x} + 3 P_i \frac{\partial u_i}{\partial x} = 0, \quad (3)
\]

Maxwell’s equation \((11-13)\)

\[
\frac{\partial}{\partial t} \left( \frac{\partial \varphi}{\partial x} \right) = n_i u_i, \quad (4)
\]

The electric field \((13)\)

\[
E = -\frac{\partial \varphi}{\partial x}, \quad (5)
\]

The mobility of ions \(\mu = \frac{u}{E}\), \(6\)

We calculate the total energy for ions according to \((12)\),

\[
\varepsilon_i = \left[ \frac{\sigma}{2} P_i + \frac{1}{2} n_i u_i^2 \right]. \quad (7)
\]

All physical quantities appearing in the above equations have been normalized by the following quantities \((3)\):

\[
\bar{u}_i \Rightarrow \frac{u_i}{C_s}, \quad \bar{n}_i \Rightarrow \frac{n_i}{n_0}, \quad \bar{x} \Rightarrow \frac{x}{\lambda_{De}}, \quad \bar{\varphi} \Rightarrow \frac{\varphi}{T_{\text{eff}}},
\]

\[
\bar{\varepsilon}_i \Rightarrow \frac{\varepsilon_i}{m_i C_s^2}, \quad \bar{P} \Rightarrow \frac{P}{P_0}, \quad \bar{t} \Rightarrow t \cdot \omega_{pi}, \quad \bar{E} \Rightarrow \frac{E e \lambda_{De}}{T_{\text{eff}}}. \quad \text{Where,}
\]

\[
C_s^2 = \frac{T_{\text{eff}}}{m_i}, \quad \omega_{pi}^2 = \frac{4 \pi e^2 n_0}{m_i}, \quad \lambda_{De}^2 = \frac{T_{\text{eff}}}{4 \pi n_0 e^2}, \quad \sigma = \frac{T_{\text{eff}}}{T_{\text{eff}}}, \quad P_0 = n_0 T_i.
\]

Now replacing the distance \(x\) and time \(t\) by a moving coordinate \(\xi = kx - \omega t\) \((8)\), where \(\omega\) and \(k = k(\omega)\) are the frequency and the wave number of the IAW. The solution is obtainable by transforming from partial to ordinary differential equations using the new variable \(\xi\), so we replace:

\[
\frac{\partial}{\partial t} \Rightarrow -\omega \frac{d}{d\xi}; \quad \frac{\partial}{\partial x} \Rightarrow k \frac{d}{d\xi}; \quad \varepsilon' = \frac{d}{d\xi} \frac{d^2}{d\xi^2}.
\]

Therefore the above equations \((1-7)\) become, where the bars are dropped,

\[
\left( u - \frac{\omega}{k} \right) n' + nu' = 0, \quad (8)
\]

\[
\left( u - \frac{\omega}{k} \right) nu' + \sigma p' + n \varphi' = 0. \quad (9)
\]
\[
\left( u - \frac{\omega}{k} \right) p' + 3pu' = 0 , \quad (10)
\]

\[
\varphi'' + \frac{1}{\omega k} nu = 0 , \quad (11)
\]

\[
E = -k\varphi' . \quad (12)
\]

The phase and group velocities are \(^{(13,14)}\):

\[
V_{ph} = \frac{\omega}{k} , \quad (13)
\]

\[
V_{gr} = \left( \frac{\partial k}{\partial \omega} \right)^{-1} . \quad (14)
\]

We solve the system of nonlinear ordinary differential equations (8-14) beginning from equations (8, 10) by the separation of variables, so

\[
\frac{n'}{n} = \frac{P'}{3P} = -\frac{u'}{u - \frac{\omega}{k}} = -A\xi^\nu , \quad (15)
\]

We substitute by \(A = -i\), \(\nu = 0\) in equation (15) to get:

\[
\frac{n'}{n} = \frac{P'}{3P} = -\frac{u'}{u - \frac{\omega}{k}} = i , \quad (16)
\]

and integrating with the conditions that at \(\xi = 0\) → \(u(0) = u_0, P(0) = P_0, \varphi(0) = \varphi_0, n(0) = n_0\) we get:

\[
P = P_0 e^{3i\xi} , \quad (17)
\]

\[
u = V_{ph} + (u_0 - V_{ph}) e^{-i\xi} , \quad (18)
\]

\[n = n_0 e^{i\xi} . \quad (19)
\]

The direct substitution and integration of equations (6, 7, 9, 12) give:

\[
\Phi = \varphi - \varphi_0 = -\frac{e^{-2i\xi}}{2k^2n_0} \left( \frac{e^{2i\xi} - 1}{k^2} \left( -u_0^2n_0 + 3e^{2i\xi} p_0\varphi + 2knu_0\omega - n_0\omega^2 \right) \right) , \quad (20)
\]

\[
E = -\frac{i e^{-2i\xi}}{kn_0} \left( k^2 \left( n_0u_0^2 - 3e^{4i\xi} p_0\varphi - 2kn_0u_0\omega + n_0\omega^2 \right) \right) , \quad (21)
\]
\[
\mu = - \frac{\imath e^{i\xi} n_0 (ku_0 + (e^{i\xi} - 1)\omega)}{k^2 (-n_0 u_0^2 + 3 e^{4i\xi} p_0 \sigma) + 2 kn_0 u_0 \omega - n_0 \omega^2},
\]
(22)  
\[
\epsilon = \frac{1}{2} \left[ \frac{e^{-i\xi} n_0 (ku_0 + (e^{i\xi} - 1)\omega)^2}{k^2} + e^{3i\xi} p_0 \sigma \right].
\]
(23)  

We substitute from equations (18, 19, 20) into equation (11) to obtain the dispersion relation which describes the nonlinear behavior of the IAW propagation.

\[
6 p_0 e^{2i\xi} \sigma - n_0 (ku_0 + (e^{i\xi} - 1)\omega) + 2 e^{-2i\xi} \left( u_0 - \frac{\omega}{k} \right)^2 = 0
\]
(24)  

Expanding the exponential functions in equation(24) with the assumption that \(|\xi|<<1\) up to the second order and replacing \(\xi\) by the original variables \(x, t\) in order to obtain the roots of the resulting equation with respect to \(\omega\) as a dispersion relation \(k=k(\omega)\) [the source case]. The proper root out of four is;

\[
k(\omega) = \frac{-c}{4xd} - \frac{s}{8dx\sqrt{3}} + \frac{s}{8dx\sqrt{3}} \]  
Where,

\[
a = 4n_0 u_0^2 - n_0 x^2 + 12 p_0 \sigma + 8i n_0 u_0 t \omega + 16 i n_0 u_0 x \omega - 24 i t p_0 \sigma \omega - 8 n_0 u_0^2 t^2 \omega^2,
\]

\[c = i n_0 u_0^2 - 3ip_0 \sigma - 2n_0 u_0^2 \omega - 2n_0 u_0 x \omega - 6 p_0 t \sigma \omega, \quad d = n_0 u_0^2 + 3 p_0 \sigma \]

\[f = -2i n_0^2 t \omega^2 + 4 n_0 \omega^3 - n_0^2 t \omega^3 + 8i n_0 t \omega^4 - 8 n_0^2 t^2 \omega^5,
\]

\[g = n_0^2 u_0 + i n_0 x \omega - 4 n_0 u_0 \omega^2 + n_0^2 t x \omega^2 - 8 n_0 t u_0 \omega^3 - 4 i n_0 x \omega^3
\]

\[+ 8 n_0 t^2 u_0 \omega^4 + 8 n_0 x t \omega^4,
\]

\[r = 2a^2 \omega^2 + 144x^2 c a g - 864x^2 d e g^2 + 1728 f x^2 c^2 \omega^2 + 576 x^2 \omega^2 d a f
\]

\[q = a^2 \omega^2 + 48x^2 c a g - 964x^2 d e g^2, \quad b = \frac{q}{r} + \sqrt{-4q^3 + r^2}
\]

\[h = \frac{x^2 q}{12\sqrt{4 a d b x^4}}, \quad s = \sqrt{-\frac{3}{4} b d \omega + 4(3c^2 \omega^2 + d \omega^2 (a - 12 d h x^2))}
\]

The disturbance of the IAWs in time at a certain location \(x\) is considered when the frequency \(\omega\) is real, while the wave number is complex, \(k = k_r (\omega) + ik_i (\omega)\) \((15)\). The phase velocity \(V_{ph}\) and damping coefficient \(\alpha\) of the corresponding waves are given by \(V_{ph} = \frac{\omega}{k_r (\omega)}\) and \(\alpha = -k_i (\omega)\).
RESULTS AND DISCUSSION

The exact solutions of the nonlinear ion-acoustic waves (IAWs) are applied to study the behavior of an unmagnetized collisionless plasma consisting of warm ions and non-thermal electrons. Our computations are performed according to the data in [11,13,14] for warm hydrogen plasma subjected to the following conditions and parameters:

\[ n_e = 10^{14} \text{ cm}^{-3} , \quad T_{\text{eff}} = 10 \text{ eV} , \quad \omega_{pi} = 1.317 \times 10^{10} \text{ rad/sec} , \quad \lambda_{De} = 2 \times 10^{-4} \text{ cm} , \quad C_s = 5.654 \times 10^{6} \text{ cm/sec}, \]

and the conditions of the system such density, pressure, potential and the flow velocity (which is the Mach number) are \( (n_0, p_0 \text{ and } u_0 = 1) \). We evaluate all variables within the nanosecond time scale at a fixed position \( x \) with the characteristic non-dimensional frequency \( \omega / \omega_{pi} << 1 \).

Our computations are carried out by the "Mathematica 5".

It is worth noting to mention some general remarks at the start, namely:

1- The estimated variables are in complex form, for a wave traveling in positive \( x \)-direction \( k_r > 0 \), the damping must be negative \( k_i < 0 \) where we take into account the parts which satisfy the physical ground governed by the strong inequality \( k \lambda_{De} << 1 \), see figures (1, 2).

2- The variables are calculated in the ranges \( \sigma = [0.0, 1.0] \) and \( t = [70, 350] \), at \( x = 4 \times 10^3 \) and \( \omega = 3.3 \times 10^{-4} \), which are controlled by the relation of \( (k- \sigma) \) and the criterion \( |\xi| << 1 \) within the dimensional nanosecond time scale \( t = [7 \times 10^{-9}, 35 \times 10^{-9}] \text{ sec} \), see figures (3).

The wave number \( k \):

From the Eq. (25) we observe that the amplitude of the wave number of IAW decreases nonlinearly as the temperature \( \sigma \) increases. The IAW can propagate only when the protons temperature is very small compared with the electrons temperature, \( T_i << T_{\text{eff}} \); otherwise they are damped, see figure(1) ; which is in a good agreement with Xue Jukui (3). We noted that the affect of nonlinearity occurs in the interval \( \sigma = [0, 0.25] \).

Figures (1-3) Plots of the real and imaginary parts of wave number \( k \) and \( \xi \) vs. \( \sigma \) and \( t \).

Figures (4-6) Plots of the pressure \( p \), density \( n \) and flow velocity \( u \) vs. \( \sigma \) and \( t \).
The pressure $P$:

It increases nonlinearly as the ions temperature $\sigma$ increases until it reaches the critical value $\sigma = \frac{T}{T_{\text{eff}}} \approx 0.25$ and then takes a constant maximum value (unity) within $\sigma = [0.25, 1.0]$. It is show that $P$ decreases slowly with time $t = [70, 350]$ for the same almost above mentioned interval of $\sigma$; see figure (4).

The density $n$:

We note that the behavior of the density coincides with that of the pressure this is in a good agreement with the physical situation. The majority of number of ions still much lower than the common density till it moving away from the critical temperature $\sigma \approx 0.25$; see figure (5).

The flow velocity $u$:

The flow velocity $u$ satisfies the requirement of being the transonic regime ($0.8 < \text{Mach number} < 1.2$) at the smaller temperature ratio $0 \leq \sigma \leq 0.23$, between $0.23 \leq \sigma \leq 0.25$ sudden nonlinear decreases then acquires a lower constant value at the rest of temperature interval, see figure (6).

The electric potential $\Phi$:

The nonlinearity is shown in the sudden break down at the critical temperature $\sigma \approx 0.25$, it is much obvious at the beginning of the time interval. It increases nonlinearly as the time increases within the interval $t = [70, 350]$ for the interval $\sigma = [0.25, 1.0]$; see figure (7).

The electric field $E$:

The electric field $E$ has the same manner as the flow velocity $u$, where it propagates in the positive $x$-direction; see figure (8).

The mobility of ions $\mu$:
The (+ve) amplitude of the mobility $\mu$ increases suddenly at $\sigma = 0.25$ for all values of $t$. It increases slowly within the interval $t = [70, 350]$ at $\sigma = [0.25, 1.0]$; see figure (9).

The total energy of ions $\varepsilon$:

The amplitude of the total energy of ions $\varepsilon$ increases as $\sigma$ increases within $\sigma = [0, 0.25]$ and decreases suddenly at $\sigma = 0.25$. The amplitude of $\varepsilon$ increases within the interval $\sigma = [0.25, 1.0]$ for all values of $t$; see figure (10). The amplitude of the ratio $\varepsilon_{(p/k)}$ (potential energy/kinetic energy) is less than unity so that the kinetic energy represents the major part of the total energy and it increases as the temperature $\sigma$ increases, see figure (11).

The velocity ratio $V (V_{ph}/V_{gr})$:

The amplitude of $V$ decreases smoothly in a linear manner until the critical temperature value and then suddenly increases to a constant value in the interval $\sigma = [0.25,1.0]$ for all the time scale; see figure (12).

CONCLUSION

In summary, we investigated the analytical solution of the nonlinear ion acoustic wave equations. These waves propagate in an unmagnetized collisionless plasma consisting of warm ions and non-thermal electrons. It is found that the relation between $(k-\sigma)$ was satisfied, where the wave number of IAW decreases nonlinearly as the temperature $\sigma$ increases. And we estimated all the characteristic variables of our system and we found that the IAWs under study propagates by transonic velocity. We tried to illustrate the behavior of the IAWs in the absence (to our knowledge) in data in the literature concerning the explicit relationships between the studied variables and $\sigma$. The analytical calculations gives that the critical value $\sigma = \frac{T_i}{T_{eff}} \approx 0.25$ distinguishes between the linear and nonlinear characters of IAW within the nanosecond time scale.

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