The magnetic field in a solenoid or Helmhotz type magnet can be represented by a Legendre function power series. Using modern computers, one can design the coils of a solenoid or Helmhotz magnet in order to produce a field with desired characteristics. The Legendre function presentation can be used with certain iron configurations as well. The design of solenoid correction coils using this technique is presented in the paper.

Introduction

The vector potential and the field inside an axisymmetric magnet (solenoid) can be expanded in a converging power series of spherical functions or Legendre functions. This can be done even when spherical and planar iron poles are present as long as the permeability of the iron is infinite. The Legendre function approach is very useful for designing solenoid magnets with a high degree of field uniformity. This approach has been programmed on the LBL 7600 computer so that one can design an axisymmetric magnet configuration which meets any desired field structure.

This paper demonstrates the following: (1) the Legendre function representation of the magnetic field generated by a single current ring, (2) the Legendre function of representation of magnetic fields in symmetrical magnets without iron, (3) the Legendre function representation of magnetic fields in symmetrical magnets with iron of infinite permeability, (4) a method for designing solenoidal coils which produce a desired field structure, and (5) examples of how Legendre functions can be used in solenoidal magnet design.

Legendre function representation of the magnetic field due to a single current loop. The Legendre function representation is a spherical coordinate representation. Therefore, the field and vector potential coordinates are given in the r, θ and φ directions (see Fig. 1). In a solenoid, one can restrict the problem further by representing the current, which is at a radius r_c, as a latitude band at an angle θ_c from the pole. The current flows in the φ direction. Thus, the vector potential A has only a φ component and the induction B has only r and θ components. In simple form, a solenoid is an axial symmetric spherical coordinate case. As a result, a Legendre function of the first kind can be used to represent the solution.

![Figure 1. A simple current loop in spherical coordinate form.](image-url)
Let us assume that there is a function $W_1$ which applies inside a sphere of radius $r_c$; and that there is a function $W_0$ which applies outside of the sphere of radius $r_c$. On the sphere, $W_1 = W_0$. These functions take the following forms:

\[ W_1 = \sum_{n=1}^{\infty} \frac{A_n}{2n(n+1)} r^n P_n(U) \]  

and

\[ W_0 = \sum_{n=1}^{\infty} \frac{A_n}{2n(n+1)} \left( \frac{1}{r} \right)^{n+1} P_n(U) \]

where $U = \cos \theta$ and where $A_n$ and $A_n^*$ are defined as follows:

\[ A_n = \mu_0 I \sin^2 \theta_c \frac{P_n^1(\cos \theta_c)}{r_c^n} \]  

and

\[ A_n^* = \mu_0 I \sin^2 \theta_c \frac{P_n^1(\cos \theta_c)}{r_c^{n+1}} \frac{2n+1}{2} \]

where $\frac{P_n^1(\cos \theta_c)}{\sin \theta_c} = \frac{\partial}{\partial \theta} P_n(\cos \theta_c)$ and where $\frac{P_n^1(\cos \theta_c)}{\cos \theta_c}$ is sometimes called a Tesselar harmonic of the first degree. The definition of symbols is as follows: $A_n$ and $A_n^*$ are power series coefficients; $\mu_0$ is the permeability of vacuum ($\mu_0 = 4\pi \times 10^{-7}$); $I$ is the current flowing in the loop (see Fig. 1); $\theta_c$ is the angle of the loop (see Fig. 1); $r_c$ is the radius of the loop (see Fig. 1); $\theta$ is the angle at which one wants to know the function $W_1$ and $W_0$; and $r$ is the radius at which one wants to know the $W$ function. Note that all dimensions are in SI units.

From the $W$ functions one can derive the vector potential $A_\phi$ and the induction components $B_r$ and $B_\theta$.  

\[ A_\phi = \frac{W}{\partial \theta} \]  

\[ B_r = \frac{\partial^2 (rw)}{\partial r^2} \]  

\[ B_\theta = \frac{\partial^2 (rw)}{\partial \theta \partial \phi} \]

Note that $A_r$, $A_\theta$ and $B_\phi$ are zero due to axial symmetry. If one does the differentiation, one finds, when $r < r_c$, that

\[ A_\phi = \sum_{n=1}^{\infty} \frac{A_n}{2n(n+1)} r^n \sin \theta \frac{P_n^1(\cos \theta)}{r_c} \]  

\[ B_r = \sum_{n=1}^{\infty} \frac{A_n}{2n} r^{n-1} \frac{P_n(\cos \theta)}{r_c} \]  

\[ B_\theta = \sum_{n=1}^{\infty} \frac{A_n}{2n} r^{n-1} \sin \theta \frac{P_n(\cos \theta)}{r_c} \]
When \( r \geq r_c \),

\[
A_\phi = \sum_{n=1}^{\infty} \frac{A_n^*}{2n(n+1)} \left( \frac{1}{r} \right)^{n+1} \sin \theta \frac{P_n^1(\cos \theta)}{}, \tag{5a}
\]

\[
B_r = \sum_{n=1}^{\infty} A_n^* \left( \frac{1}{r} \right)^{n+2} P_n(\cos \theta), \tag{5b}
\]

\[
B_\theta = \sum_{n=1}^{\infty} \frac{A_n^*}{2n} \left( \frac{1}{r} \right)^{n+1} \sin \theta \frac{P_n^1(\cos \theta)}{}. \tag{5c}
\]

Equations 4 and 5 apply for a single current loop. If a number of current loops make up a solenoid magnet configuration, a simple sum of equations 4 or 5 can be used to calculate \( B_r \), \( B_\theta \) and \( A_\phi \) provided the loop centers all lie on the same line and that line is perpendicular to a plane formed by those circular loops. Thus, an axially symmetric solenoid (where the axis of all loops lie on the same center line) can be made up of many loops of varying radii and varying distances from a point on the center line called the origin.

An axially symmetric solenoid (where the axis of all loops lie on the same center line) can be made up of many loops of varying radii and varying distances from a point on the center line called the origin.

A solenoid magnet can be divided into many current bands which are at various radii \( r_c \) and various angles \( \theta_c \). The field generated by such a magnet can be calculated by summing the fields generated by the individual currents. Reasonably accurate calculations (say to 1 part in \( 10^2 \)) of the solenoid magnet field can be achieved by dividing the coil into a hundred parts. One can simplify the calculation of the solenoid field, if the magnet is symmetric about the \( \theta = \pi/2 \) plane as well as the axis (\( \theta = 0 \) or \( \theta = \pi \)). The symmetrical solenoid case is discussed in the next section.

Symmetrical solenoid magnets without iron. Two kinds of symmetry will be described here. The first, which one may call dipole symmetry, has current of the same sign which are symmetrical about the \( \theta = \pi/2 \) plane (see Fig. 2a). The second kind of symmetry, which can be called quadrupole symmetry, has currents of opposite sign which are symmetrical about the \( \theta = \pi/2 \) plane (see Fig. 2b). This permits one to treat a current at \( \theta = \theta_c \) along with a current at \( \theta = \pi-\theta_c \). As a result, one can reduce the number of calculation steps by a factor of two or more.

Figure 2. Symmetrical magnet structures with two current loops.
Before proceeding with the symmetrical air core magnet case, it is useful to look at the properties of Legendre functions of first order and first degree.

When \( n \) is odd,
\[
P_n^1(\cos\theta) = P_n^1(\cos(\pi - \theta)) ,
\]
(6a)

and when \( n \) is even,
\[
P_n^1(\cos\theta) = -P_n^1(\cos(\pi - \theta)) .
\]
(6b)

From equations 6a and 6b one can show that for a cylindrical solenoid with symmetry, when \( r > r_c \),
\[
A_\phi = \sum_{n=1}^{\infty} \frac{A_n}{n(n+1)} r^n \sin\theta P_n^1(\cos\theta) ,
\]
(7a)
\[
P_r = \sum_{n=1}^{\infty} A_n r^{n-1} P_n(\cos\theta) ,
\]
(7b)
\[
P_\theta = \sum_{n=1}^{\infty} \frac{A_n}{n} r^{n-1} \sin\theta P_n^1(\cos\theta) .
\]
(7c)

When \( r > r_c \),
\[
A_\phi = \sum_{n=1}^{\infty} \frac{A_n^*}{n(n+1)} \left(\frac{1}{r}\right)^{n+1} \sin\theta P_n^1(\cos\theta) ,
\]
(7d)
\[
P_r = \sum_{n=1}^{\infty} A_n^* \left(\frac{1}{r}\right)^{n+2} P_n(\cos\theta) ,
\]
(7e)
\[
P_\theta = \sum_{n=1}^{\infty} \frac{A_n^*}{n} \left(\frac{1}{r}\right)^{n+2} \sin\theta P_n^1(\cos\theta) .
\]
(7f)

The \( A_n \) and \( A_n^* \) terms are defined by equations 2a and 2b. The values of \( A_n \) and \( A_n^* \) given by equations 2a and 2b do not apply for all values of \( n \) when there is dipole symmetry, \( A_n \) and \( A_n^* \) are equal to zero for all even \( n \). The odd values of \( A_n \) and \( A_n^* \) are given by equations 2a and 2b. When there is quadrupole symmetry, \( A_n^* \) and \( A_n^* \) are equal to zero for all odd \( n \). The even values of \( A_n \) and \( A_n^* \) are given by equations 2a and 2b.

Most simple solenoids and Helmholtz coils are of dipole symmetry. There are coils such as the Lawrence Livermore Laboratory baseball coils which exhibit quadrupole symmetry. In any event, the concept of magnetic symmetry is very important for many cases found in the real world. The use of symmetry permits one to make a substantial reduction in the number of calculations performed.

Symmetrical solenoid magnets with iron. Expansion of the magnetic field with Legendre functions may be applied to solenoidal (axially symmetric) magnets which have iron shields. One must assume that the permeability of the iron \( \mu \) is infinite. (It should be noted that the real iron case \( \mu \neq \infty \) can often be treated as a perturbation of the \( \mu = \infty \) case.) Two kinds of iron poles are assumed here: (1) spherical iron, and (2) flat poles which are perpendicular to the magnet axis. In both cases, the method of images is employed in the calculations.

Before proceeding with the iron cases, it is useful to develop a more general power series expansion which may be applied inside the innermost coil of the solenoid (where \( r \) will always be less than \( r_c \)). Symmetry is assumed to be used in the calculation. The series form for vector potential \( A_\phi \) and induction \( B_r \) and \( B_\theta \) is as follows:
\[
A_\phi = \sum_{n=1}^{\infty} \frac{c_n}{n(n+1)} r^n \sin\theta P_n^1(\cos\theta) ,
\]
(8a)
Equations 8a, 8b, and 8c can be used for symmetric magnets of either type since the $C_n$ component is a function of $r_c$, $B_C$ and $I$ only. The $C_n$ components can be divided into two parts as follows:

$$C_n = A_n + B_n$$

There the $A_n$ component is due to the coils alone and the $B_n$ component is due to the image currents in the iron. The $A_n$ terms are given by equation 2a; the $B_n$ terms will be given in this section.

The $B_n$ terms in a symmetrical magnet behave in the same way as the $A_n$ terms. When dipole symmetry is evoked, $B_n = 0$ when $n$ is even; when quadrupole symmetry is evoked, $B_n = 0$ when $n$ is odd. The $B_n$ terms depend on the geometry of the iron shield. (The shield geometry affects the location of the image currents.) Two infinite $\mu$ iron geometries are of general interest; they are: (1) A spherical shield where the coil is inside a hollow iron sphere. (2) Two flat parallel pole pieces which lie parallel to the $\theta = \pi/2$ plane. The first geometry may be of some interest even when the central induction of the solenoid exceeds 2 tesla. The latter geometry is a reasonable model for the iron shields used in a solenoid which has a central induction of less than say 1.8 to 2.0 tesla.

(a) A spherical iron shield. This case has only one image current for each coil current. The image current radius $r_{c4} = R^2/r_c$ where $R$ is the iron shield radius and $r_c$ is the coil current radius. The image current angle $\theta_c$ is the same as the coil current angle $\theta_c$. (See Fig. 3.) Therefore, the $B_n$ for the spherical iron is as follows:

$$B_n = \mu_0 I \sin^2 \theta_c \frac{1}{r_c} \frac{C_n}{R^{2n}}$$

The use of a spherical iron shell is desirable when the central induction of the solenoid exceeds two tesla. The shield radius $R$ can be adjusted so that the iron is not saturated. If the shield is allowed to saturate, the effects of symmetry will remain. Saturation will appear as perturbations to the $B_n$ terms. The lowest perturbation term will appear first and the others will appear as the iron saturates. Some of the iron saturation perturbation terms can be controlled by shaping the outer boundary of the iron shield. Spherical iron shields should be usable even when the central induction of the solenoid approaches four tesla.

Figure 3. A spherical iron shield around a symmetrical magnet (dipole symmetry) showing image currents.
(b) Parallel iron pole pieces. The \( B_n \) term will consist of an infinite number of image currents. The first image current will be located the same distance inside the iron pole piece as the current is from the pole. An image current is located inside the opposite pole as well. The image currents also form image currents in the poles (see Fig. 4). In general, the \( B_n \) term in a symmetrical magnet will take the following form:

\[
B_n = \sum_{m=1}^{\infty} b_m \mu_0 I \sin^2(\theta_{cm}) P^1_n(\cos\theta_{cm}) \left( \frac{1}{r_{cm}} \right)^n.
\]

The value of \( b_m \) depends on the symmetry used. \( b_m = 1 \) for all \( m \) when dipole symmetry applies. When quadrupole symmetry is used \( b_m = +1 \) when \( m=1, 4, 5, 8, 9, \ldots \) and \( b_m = -1 \) when \( m=2, 3, 6, 7, 10, 11, \ldots \).

\( \theta_{cm} \) and \( r_{cm} \) take the following form when \( m \) is odd:

\[
\theta_{cm} = \tan^{-1} \left( \frac{2r_c \sin \theta_c}{(m+1)L-2r_c \cos \theta_c} \right)
\]

\[
r_{cm} = \left[ \left( \frac{(m+1)}{2} L - r_c \cos \theta_c \right)^2 + r_c^2 \sin^2 \theta_c \right]^{1/2}.
\]

When \( m \) is even:

\[
\theta_{cm} = \tan^{-1} \left( \frac{2r_c \sin \theta_c}{mL-2r_c \cos \theta_c} \right),
\]

\[
r_{cm} = \left[ \left( \frac{m}{2} L + r_c \cos \theta_c \right)^2 + r_c^2 \sin^2 \theta_c \right]^{1/2}.
\]

where \( \theta_c \) is the coil current angle; \( r_c \) is the coil current radius \( (m) \); \( I \) is the magnitude of the current \( (A) \); \( L \) is the distance between the flat parallel iron pole pieces \( (m) \); and \( \mu_0 \) is the permeability of a vacuum.

The use of parallel poles is particularly desirable when the central induction of the solenoid is below the saturation induction of the iron. Saturation of the iron introduces field aberrations in which the lowest order term will be worst, followed by the next order term. If the solenoid has many evenly spaced coils, the infinite solenoid case is approximated. Using the parallel pole iron shield, one can get a field which is a good dipole type structure.

![Figure 4. A symmetrical magnet (dipole symmetry) with flat iron poles.](image)
A method of solenoid magnet design. Axisymmetric magnets can be designed with any desired field structure as long as physical law is not broken. Let us suppose that a given solenoidal magnet has k parameters G(1), G(2)...G(k), which can be varied to produce a field of desired structure. Using these parameters one can, in theory, design a current distribution which has k terms of the C_n power series coefficients (where C_n is defined for all n by equation 9) tailored to their desired values.

In general, one would choose the lowest values of C_n because these would affect the magnetic field the most. For example, if the magnet is not to have symmetry, one would choose C_1, C_2...C_k as the power series coefficients to tailor. If the magnet is to have dipole symmetry, C_1, C_3, C_5...C_{2k-1} would be chosen. If the magnet is to have quadrupole symmetry, C_2, C_4, C_6...C_{2k} would be chosen.

The method used by our computer program is a Newton method. First guess values of G(1)...G(k) are chosen. The structure of the field (the various C_n coefficients) is calculated. Then the first derivative of these coefficients with respect to each of the G parameters is calculated. Using these derivatives, one can set up the linear simultaneous equations which find correction functions X(1)...X(k) for the various G parameters which will produce the desired magnetic structure as represented by the coefficients D_1, D_2...D_k. The simultaneous equations take the following form:

\[
\begin{align*}
\frac{3C_1}{3G(1)} X(1) + \frac{3C_1}{3G(2)} X(2) + \ldots + \frac{3C_1}{3G(k)} X(k) &= D_1 \\
\frac{3C_2}{3G(1)} X(1) + \frac{3C_2}{3G(2)} X(2) + \ldots + \frac{3C_2}{3G(k)} X(k) &= D_2 \\
\vdots & \quad \vdots \\
\frac{3C_n}{3G(1)} X(1) + \frac{3C_n}{3G(2)} X(2) + \ldots + \frac{3C_n}{3G(k)} X(k) &= D_n \\
\end{align*}
\]  

(12)

The preceding equations are solved for the correction function X by matrix inversion. The correction functions are added to the G functions such that

\[
G_{\text{new}} = G_{\text{old}} + X.
\]  

(13)

One calculates the various power series coefficients C_1, C_2...C_k using the new values of G. The new values of C_1, C_2...C_k are compared to D_1, D_2...D_k on a term-by-term basis. If D-C<ε (where ε is a small number like 10^-5) for all values of C and D from n=1 to n=k, a solution has been found. If D-C > ε, derivatives of C with respect to the new values of G are taken and the simultaneous equations (Eq. 12) are solved so new values of G are found. Convergence, if there is convergence, will usually take less than ten iterations.

A lack of convergence is usually caused by the following: (1) there is no solution within the parameter boundary chosen which yields the desired magnetic structure, or, (2) the first guess was incorrectly chosen. Convergence may be obtained on solutions which may be correct mathematically, but they have no physical meaning. It may take several tries to find a current geometry which converges to a mathematically correct solution which has physical meaning. The two sample problems given in the next section illustrate the way the technique works.

Sample problems. The utility of the Legendre function power series technique is illustrated by the sample solutions given in this section. These problems are: (1) a uniform air core spherical solenoid and (2) correction coils for a solenoid with flat poles. Both of these problems could be encountered in the real world.

(a) Uniform field air core spherical solenoid. The uniform field air core spherical solenoid problem could be encountered in the field of solid state physics. Solenoid magnets have been built with a very uniform field within a small sphere located at the center of the magnet. Let us postulate a solenoid with the parameters given in Table 1.
Table 1. Basic parameters for a uniform field air core solenoid magnet.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Induction at center</td>
<td>3.0 tesla</td>
</tr>
<tr>
<td>Solenoid diameter</td>
<td>0.3 m</td>
</tr>
<tr>
<td>Solenoid length</td>
<td>0.3 m</td>
</tr>
<tr>
<td>Field quality within a sphere</td>
<td>10⁻⁶</td>
</tr>
<tr>
<td>High quality field sphere diameter</td>
<td>0.1 m</td>
</tr>
<tr>
<td>Coil maximum thickness</td>
<td>0.01 m</td>
</tr>
</tbody>
</table>

The spherical solenoid consists of three coil blocks in each half (see Fig. 5). The magnet coils have dipole symmetry. So, \( C_n = 2, 4, 6 \ldots \) are eliminated by symmetry. There are five coil parameters available to zero five \( C_n \) coefficients. They are the five block angles \( G(2) \) through \( G(6) \) shown in Fig. 5. These five block angles can be set so that \( C_n \approx 0 \), for \( n = 3, 5, 7, 9 \) and 11; and \( C_n \neq 0 \) when \( n > 13 \). Figure 5 shows the five coil parameters which are varied to eliminate the \( C_3, C_5 \ldots \) to \( C_{11} \). A sixth parameter \( G(1) \), the current density in the coil blocks (which are all assumed to have the same current density), determines the induction of the magnet at its center (\( C_1 = 3.0 \) T). Table 2 shows the first guess and final value of each of the six magnet parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>First Guess</th>
<th>Final Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 ) The coil current density ( \left( \text{Am}^2 \right) )</td>
<td>( 3 \times 10^8 )</td>
<td>( 3.516 \times 10^8 )</td>
</tr>
<tr>
<td>( G_2 ) 2nd coil angle, Block 1 (deg.)</td>
<td>60.00</td>
<td>58.3490</td>
</tr>
<tr>
<td>( G_3 ) 1st coil angle, Block 2 (deg.)</td>
<td>50.00</td>
<td>55.1390</td>
</tr>
<tr>
<td>( G_4 ) 2nd coil angle, Block 2 (deg.)</td>
<td>45.00</td>
<td>39.2733</td>
</tr>
<tr>
<td>( G_5 ) 1st coil angle, Block 3 (deg.)</td>
<td>30.00</td>
<td>30.3888</td>
</tr>
<tr>
<td>( G_6 ) 2nd coil angle, Block 3 (deg.)</td>
<td>25.00</td>
<td>21.0544</td>
</tr>
</tbody>
</table>

Figure 5. A good field quality spherical solenoid.
Table 3 shows a value of $C_n$ at the surface of 0.1-m diameter sphere for each $n$ between 1 and 20. The table shows the value for the first guess case and the final case.

<table>
<thead>
<tr>
<th>$n$</th>
<th>First Guess</th>
<th>Final Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.25928</td>
<td>3.00000</td>
</tr>
<tr>
<td>3</td>
<td>$-7.347 \times 10^{-2}$</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>$-3.107 \times 10^{-4}$</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>$2.967 \times 10^{-4}$</td>
<td>-</td>
</tr>
<tr>
<td>9</td>
<td>$-5.671 \times 10^{-6}$</td>
<td>-</td>
</tr>
<tr>
<td>11</td>
<td>$-1.929 \times 10^{-6}$</td>
<td>-</td>
</tr>
<tr>
<td>13</td>
<td>$-1.456 \times 10^{-6}$</td>
<td>$-1.849 \times 10^{-6}$</td>
</tr>
<tr>
<td>15</td>
<td>$-2.036 \times 10^{-8}$</td>
<td>$-1.499 \times 10^{-7}$</td>
</tr>
<tr>
<td>17</td>
<td>$2.914 \times 10^{-8}$</td>
<td>$1.943 \times 10^{-8}$</td>
</tr>
<tr>
<td>19</td>
<td>$4.611 \times 10^{-10}$</td>
<td>$2.663 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

*Note $C_n = 0$ when $n = 2, 4, 6, ... 20$.

The solution given in Tables 2 and 3 can be obtained theoretically in a superconducting solenoid. It should be pointed out that real-life superconducting magnets have finite size winding errors. It might be possible to build a real magnet which has a field uniformity of $10^{-5}$ or better. It should be noted that diamagnetic current will cause large distortions in the field. As a result, field uniformities better than $10^{-7}$ may be difficult to obtain in real life without the use of special correction winding to get rid of the effects of residual field due to diamagnetic currents.

(b) Correction coils for a magnet with flat poles. Correction coils can be designed so that they produce $C_n$ coefficients of a certain magnitude. There is a need for such correction coils in some of the solenoids now being designed for use in colliding beam experiments. These magnets have superconducting coils which are bounded at the ends by flat iron poles. The solenoid is supposed to produce a perfectly uniform field. Misalignment, saturation of the iron poles, and other effects may introduce errors in the field of a part or two in a thousand. Correction coils can remove 90 percent of these errors.

Figure 6 shows a system of eight coils which are to produce field corrections. The coils, depending on how they are powered are supposed to produce $C_1, C_2, C_3, C_4, C_5, C_6$, and $C_7$ while eliminating all other $C_n$ below $n = 9$. The eight coils are symmetrically placed on either side of the midplane, and evenly spaced between the iron poles ($\mu = \infty$) which are infinite planes (see Fig. 6). Table 4 gives the basic parameters of the correction coil blocks.

<table>
<thead>
<tr>
<th>Table 4. Basic parameters of the correction coils.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correction coil spacing</td>
</tr>
<tr>
<td>Iron pole spacing</td>
</tr>
<tr>
<td>Correction coil radius</td>
</tr>
<tr>
<td>Value of $C_n$ desired at surface of reference sphere</td>
</tr>
<tr>
<td>Radius of reference sphere</td>
</tr>
</tbody>
</table>
Figure 6. A correction coil problem.

Symmetry is used to do the calculation. Dipole symmetry is used for \( C_1, C_3, C_5 \) and \( C_7 \) and quadrupole symmetry is used for \( C_2, C_4 \) and \( C_6 \). The parameter which one solves for is the current in each of the coils. There is a different current distribution in each of the coils for each correction \( C_n \) produced. The simultaneous equations generated by the computer are linear, so no iteration is required.

Table 5 shows current in each of the eight coils when the desired coefficient is produced. Table 6 shows the values of other coefficients when the desired one is produced.

Table 5. The current in the eight correction coils when a desired multipole is generated on the surface of a 1.0 m radius reference sphere.

<table>
<thead>
<tr>
<th>Coil</th>
<th>Current (A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>n = 1</td>
<td>n = 2</td>
</tr>
<tr>
<td>Coil 1</td>
<td>1552</td>
</tr>
<tr>
<td>Coil 2</td>
<td>1640</td>
</tr>
<tr>
<td>Coil 3</td>
<td>1572</td>
</tr>
<tr>
<td>Coil 4</td>
<td>1596</td>
</tr>
<tr>
<td>Coil 5</td>
<td>1596</td>
</tr>
<tr>
<td>Coil 6</td>
<td>1572</td>
</tr>
<tr>
<td>Coil 7</td>
<td>1640</td>
</tr>
<tr>
<td>Coil 8</td>
<td>1552</td>
</tr>
</tbody>
</table>

Desired \( C_n + \frac{a}{n} \):
- \( 0.005 \), \( 0.005 \), \( 0.005 \), \( 0.005 \), \( 0.005 \), \( 0.005 \), \( 0.005 \)

A correction coil scheme similar to the one shown here is expected to be used on the TPC solenoid which is to be built in 1978. This magnet is supposed to produce a magnetic field which is good to a couple of parts in ten thousand. In the real TPC magnet, the iron poles are not infinite planes of infinite permeability. A technique such as this is expected to provide first order correction.
Table 6. The values of various $C_n$ coefficients when the desired correction $C_n$ is turned on.

<table>
<thead>
<tr>
<th>n</th>
<th>$C_{nD} = 1$</th>
<th>$C_{nD} = 2$</th>
<th>$C_{nD} = 3$</th>
<th>$C_{nD} = 4$</th>
<th>$C_{nD} = 5$</th>
<th>$C_{nD} = 6$</th>
<th>$C_{nD} = 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00500</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0.00500</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>0.00500</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>0.00500</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.00500</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.00500</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.00500</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>-0.00003</td>
<td></td>
<td>-0.00020</td>
<td></td>
<td>-0.00128</td>
<td></td>
<td>-0.00258</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>-0.00009</td>
<td></td>
<td>-0.00092</td>
<td></td>
<td>-0.00252</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0.00009</td>
<td></td>
<td>-0.00020</td>
<td></td>
<td>0.00014</td>
<td></td>
<td>-0.00117</td>
</tr>
<tr>
<td>12</td>
<td></td>
<td>-0.00001</td>
<td></td>
<td>0.00064</td>
<td></td>
<td>0.00091</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>-0.00017</td>
<td></td>
<td>0.00039</td>
<td></td>
<td>0.03007</td>
<td></td>
<td>0.00174</td>
</tr>
</tbody>
</table>

*On the surface of a reference sphere 1.0 m in radius

References


When \( r > r_c \),

\[
A_\phi = \sum_{n=1}^{\infty} A_n \frac{e^{i n \theta}}{2n+1 \cdot (2n+1)} \left( \frac{1}{r} \right)^{n+1} \sin \theta P_n^1(\cos \theta),
\]

(5a)

\[
B_r = \sum_{n=1}^{\infty} A_n \frac{e^{i n \theta}}{2n+1 \cdot (2n+1)} \left( \frac{1}{r} \right)^{n+2} P_n(\cos \theta),
\]

(5b)

\[
B_\theta = \sum_{n=1}^{\infty} A_n \frac{e^{i n \theta}}{2n+1 \cdot (2n+1)} \left( \frac{1}{r} \right)^{n+1} \sin \theta P_n^1(\cos \theta).
\]

(5c)

Equations 4 and 5 apply for a single current loop. If a number of current loops make up a solenoid magnet configuration, a simple sum of equations 4 or 5 can be used to calculate \( B_r \), \( B_\theta \) and \( A_\phi \) provided the loop centers all lie on the same line and that line is perpendicular to a plane formed by those circular loops. Thus, an axial symmetric solenoid (where the axis of all loops lie on the same center line) can be made up of many loops of varying radii and varying distances from a point on the center line called the origin.

A solenoid magnet can be divided into many current bands which are at various radii \( r_c \) and various angles \( \theta_c \). The field generated by such a magnet can be calculated by summing the fields generated by the individual currents. Reasonably accurate calculations (say to 1 part in \( 10^3 \)) of the solenoid magnet field can be achieved by dividing the coil into a hundred parts. One can simplify the calculation of the solenoid field, if the magnet is symmetric about the \( \theta = \pi/2 \) plane as well as the axis (\( \phi = 0 \) or \( \theta = \pi \)). The symmetrical solenoid case is discussed in the next section.

Symmetrical solenoid magnets without iron. Two kinds of symmetry will be described here. The first, which one may call dipole symmetry, has current of the same sign which are symmetrical about the \( \theta = \pi/2 \) plane (see Fig. 2A). The second kind of symmetry, which can be called quadrupole symmetry, has currents of opposite sign which are symmetrical about the \( \theta = \pi/2 \) plane (see Fig. 2b). This permits one to treat a current at \( \theta = \theta_c \) along with a current at \( \theta = \pi - \theta_c \). As a result, one can reduce the number of calculation steps by a factor of two or more.

Figure 2. Symmetrical magnet structures with two current loops.
Before proceeding with the symmetrical air core magnet case, it is useful to look at the properties of Legendre functions of first order and first degree.

When \( n \) is odd, \[ P_n^1(\cos \theta) = P_n^1(\cos[n-\theta]) \] (6a)

and when \( n \) is even, \[ P_n^1(\cos \theta) = -P_n^1(\cos[n-\theta]) \] (6b)

From equations 6a and 6b one can show that for a cylindrical solenoid with symmetry, when \( r > r_c \):

\[
A_\phi = \sum_{n=1}^{\infty} \frac{A_n}{n(n+1)} r^n \sin \theta P_n^1(\cos \theta),
\]

(7a)

\[
B_r = \sum_{n=1}^{\infty} A_n r^{n-1} P_n(\cos \theta),
\]

(7b)

\[
B_\theta = \sum_{n=1}^{\infty} A_n r^{n-1} \sin \theta P_n^1(\cos \theta).
\]

(7c)

When \( r > r_c \):

\[
A_\phi = \sum_{n=1}^{\infty} \frac{A_n^{\star}}{n(n+1)} \left( \frac{1}{r} \right)^{n+1} \sin \theta P_n^1(\cos \theta),
\]

(7d)

\[
B_r = \sum_{n=1}^{\infty} A_n^{\star} \left( \frac{1}{r} \right)^{n+2} P_n(\cos \theta),
\]

(7e)

\[
B_\theta = \sum_{n=1}^{\infty} A_n^{\star} \left( \frac{1}{r} \right)^{n+2} \sin \theta P_n^1(\cos \theta).
\]

(7f)

The \( A_n \) and \( A_n^{\star} \) terms are defined by equations 2a and 2b. The values of \( A_n \) and \( A_n^{\star} \) given by equations 2a and 2b do not apply for all values of \( n \) when there is dipole symmetry. \( A_n \) and \( A_n^{\star} \) are equal to zero for all even \( n \). The odd values of \( A_n \) and \( A_n^{\star} \) are given by equations 2a and 2b. When there is quadrupole symmetry, \( A_n \) and \( A_n^{\star} \) are equal to zero for all odd \( n \). The even values of \( A_n \) and \( A_n^{\star} \) are given by equations 2a and 2b.

Most simple solenoids and Helmholtz coils are of dipole symmetry. There are coils such as the Lawrence Livermore Laboratory baseball coils which exhibit quadrupole symmetry. In any event, the concept of magnetic symmetry is very important for many cases found in the real world. The use of symmetry permits one to make a substantial reduction in the number of calculations performed.

**Symmetrical solenoid magnets with iron.** Expansion of the magnetic field with Legendre functions may be applied to solenoidal (axially symmetric) magnets which have iron shields. One must assume that the permeability of the iron \( \mu \) is infinite. (It should be noted that the real iron case \( \mu \neq \infty \) can often be treated as a perturbation of the \( \mu = \infty \) case.) Two kinds of iron poles are assumed here: (1) spherical iron, and (2) flat poles which are perpendicular to the magnet axis. In both cases, the method of images is employed in the calculations.

Before proceeding with the iron cases, it is useful to develop a more general power series expansion which can be applied inside the innermost coil of the solenoid (where \( r \) will always be less than \( r_c \)). Symmetry is assumed to be used in the calculation. The series form for vector potential \( A_\phi \) and induction \( B_r \) and \( B_\theta \) is as follows:

\[
A_\phi = \sum_{n=1}^{\infty} \frac{C_n}{n(n+1)} r^n \sin \theta P_n^1(\cos \theta),
\]

(8a)
Equations 8a, 8b, and 8c can be used for symmetrical magnets of either type since the $C_n$ component is a function of $r_c$, $\theta_c$ and $I$ only. The $C_n$ components can be divided into two parts as follows:

$$C_n = A_n + B_n$$  \hspace{1cm} (9)

There the $A_n$ component is due to the coils alone and the $B_n$ component is due to the image currents in the iron. The $A_n$ terms are given by equation 2a; the $B_n$ terms will be given in this section.

The $B_n$ terms in a symmetrical magnet behave in the same way as the $A_n$ terms. When dipole symmetry is evoked, $B_n = 0$ when $n$ is even; when quadrupole symmetry is evoked, $B_n = 0$ when $n$ is odd. The $B_n$ terms depend on the geometry of the iron shield. (The shield geometry affects the location of the image currents.) Two infinite $\mu$ iron geometries are of general interest; they are: (1) a spherical shield where the coil is inside a hollow iron sphere. (2) Two flat parallel pole pieces which lie parallel to the $\theta = \pi/2$ plane. The first geometry may be of some interest even when the central induction of the solenoid exceeds 2 tesla. The latter geometry is a reasonable model for the iron shields used in a solenoid which has a central induction of less than say 1.8 to 2.0 tesla.

(a) A spherical iron shield. This case has only one image current for each coil current. The image current radius $r_{ci} = R/r_c$ where $R$ is the iron shield radius and $r_c$ is the coil current radius. The image current angle $\theta_c$ is the same as the coil current angle $\theta_c$. (See Fig. 3.) Therefore, the $B_n$ for the spherical iron is as follows:

$$B_n = \mu_0 \int \sin^2 \theta_c p_n L_n \cos \theta_c \frac{r^n}{R^2}$$  \hspace{1cm} (10)

The use of a spherical iron shell is desirable when the central induction of the solenoid exceeds two tesla. The shield radius $R$ can be adjusted so that the iron is not saturated. If the shield is allowed to saturate, the effects of symmetry will remain. Saturation will appear as perturbations to the $B_n$ terms. The lowest perturbation term will appear first and the others will appear as the iron saturates. Some of the iron saturation perturbation terms can be controlled by shaping the outer boundary of the iron shield. Spherical iron shields should be usable even when the central induction of the solenoid approaches four tesla.

Figure 3. A spherical iron shield around a symmetrical magnet (dipole symmetry) showing image currents.
(b) Parallel iron pole pieces. The \( B_n \) term will consist of an infinite number of image currents. The first image current will be located the same distance inside the iron pole piece as the current is from the pole. An image current is located inside the opposite pole as well. The image currents also form image currents in the poles (see Fig. 4). In general, the \( B_n \) term in a symmetrical magnet will take the following form:

\[
B_n = \sum_{m=1}^{\infty} b_m \mu_0 I \sin^2(\theta_{cm}) \frac{p_n^2(\cos \theta_{cm})}{r_{cm}}.
\]  

(11)

The value of \( b_m \) depends on the symmetry used. \( b_m = 1 \) for all \( m \) when dipole symmetry applies. When quadrupole symmetry is used \( b_m = 1 \) when \( m = 1, 4, 5, 8, 9, \ldots \) and \( b_m = -1 \) when \( m = 2, 3, 6, 7, 10, 11, \ldots \).

\[ \theta_{cm} \text{ and } r_{cm} \text{ take the following form when } m \text{ is odd:} \]

\[
\theta_{cm} = \tan^{-1} \left( \frac{2r_c \sin \theta_c}{(m+1)L - 2r_c \cos \theta_c} \right)
\]

(11a)

\[
r_{cm} = \left[ \left( \frac{(m+1)}{2} L - r_c \cos \theta_c \right)^2 + r_c^2 \sin^2 \theta_c \right]^{1/2}.
\]

(11b)

When \( m \) is even:

\[
\theta_{cm} = \tan^{-1} \left( \frac{2r_c \sin \theta_c}{mL - 2r_c \cos \theta_c} \right),
\]

(11c)

\[
r_{cm} = \left[ \left( \frac{m}{2} L + r_c \cos \theta_c \right)^2 + r_c^2 \sin^2 \theta_c \right]^{1/2}.
\]

(11d)

where \( \theta_c \) is the coil current angle; \( r_c \) is the coil current radius (m); I is the magnitude of the current (A); \( L \) is the distance between the flat parallel iron pole pieces (m); and \( \mu_0 \) is the permeability of a vacuum.

The use of parallel poles is particularly desirable when the central induction of the solenoid is below the saturation induction of the iron. Saturation of the iron introduces field aberrations in which the lowest order term will be worst, followed by the next order term. If the solenoid has many evenly spaced coils, the infinite solenoid case is approximated. Using the parallel pole iron shield, one can get a field which is a good dipole type structure.

![Figure 4. A symmetrical magnet (dipole symmetry) with flat iron poles.](image)
A method of solenoid magnet design. Axisymmetric magnets can be designed with any desired field structure as long as physical law is not broken. Let us suppose that a given solenoidal magnet has \( k \) parameters \( G(1), G(2) \ldots G(k) \), which can be varied to produce a field of desired structure. Using these parameters one can, in theory, design a current distribution which has \( k \) terms of the \( C_n \) power series coefficients (where \( C_n \) is defined for all \( n \) by equation 9) tailored to their desired values.

In general, one would choose the lowest values of \( C_n \) because these would affect the magnetic field the most. For example, if the magnet is not to have symmetry, one would choose \( C_1, C_2 \ldots C_k \) as the power series coefficients to tailor. If the magnet is to have dipole symmetry, \( C_1, C_3, C_5 \ldots C_{2k-1} \) would be chosen. If the magnet is to have quadrupole symmetry, \( C_2, C_4, C_6 \ldots C_{2k} \) would be chosen.

The method used by our computer program is a Newton method. First guess values of \( G(1) \ldots G(k) \) are chosen. The structure of the field (the various \( C_n \) coefficients) is calculated. Then the first derivative of these coefficients with respect to each of the \( G \) parameters is calculated. Using these derivatives, one can set up the linear simultaneous equations which find correction functions \( X(1) \ldots X(k) \) for the various \( G \) parameters which will produce the desired magnetic structure as represented by the coefficients \( D_1, D_2 \ldots D_k \). The simultaneous equations take the following form:

\[
\begin{align*}
\frac{\partial C_1}{\partial G(1)} X(1) + \frac{\partial C_1}{\partial G(2)} X(2) + \ldots + \frac{\partial C_1}{\partial G(k)} X(k) &= D_1 \\
\frac{\partial C_2}{\partial G(1)} X(1) + \frac{\partial C_2}{\partial G(2)} X(2) + \ldots + \frac{\partial C_2}{\partial G(k)} X(k) &= D_2 \\
\vdots \\
\frac{\partial C_k}{\partial G(1)} X(1) + \frac{\partial C_k}{\partial G(2)} X(2) + \ldots + \frac{\partial C_k}{\partial G(k)} X(k) &= D_k
\end{align*}
\]

The preceding equations are solved for the correction function \( X \) by matrix inversion. The correction functions are added to the \( G \) functions such that

\[
G_{\text{new}} = G_{\text{old}} + X
\]

One calculates the various power series coefficients \( C_1, C_2 \ldots C_k \) using the new values of \( G \). The new values of \( C_1, C_2 \ldots C_k \) are compared to \( D_1, D_2 \ldots D_k \) on a term-by-term basis. If \( D-C < \epsilon \) (where \( \epsilon \) is a small number like \( 10^{-6} \)) for all values of \( n \) from 1 to \( n \), a solution has been found. If \( D-C > \epsilon \), derivatives of \( C \) with respect to the new values of \( G \) are taken and the simultaneous equations (Eq. 12) are solved so new values of \( G \) are found. Convergence, if there is convergence, will usually take less than ten iterations.

A lack of convergence is usually caused by the following: (1) there is no solution within the parameter boundary chosen which yields the desired magnetic structure, or, (2) the first guess was incorrectly chosen. Convergence may be obtained on solutions which may be correct mathematically, but they have no physical meaning. It may take several tries to find a current geometry which converges to a mathematically correct solution which has physical meaning. The two sample problems given in the next section illustrate the way the technique works.

Sample problems. The utility of the Legendre function power series technique is illustrated by the sample solutions given in this section. These problems are: (1) a uniform air core spherical solenoid and (2) correction coils for a solenoid with flat poles. Both of these problems could be encountered in the real world.

(a) Uniform field air core spherical solenoid. The uniform field air core spherical solenoid problem could be encountered in the field of solid state physics. Solenoid magnets have been built with a very uniform field within a small sphere located at the center of the magnet. Let us postulate a solenoid with the parameters given in Table 1.
Table 1. Basic parameters for a uniform field air-core solenoid magnet.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Induction at center</td>
<td>3.0 tesla</td>
</tr>
<tr>
<td>Solenoid diameter</td>
<td>0.3 m</td>
</tr>
<tr>
<td>Solenoid length</td>
<td>0.3 m</td>
</tr>
<tr>
<td>Field quality within a sphere</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>High quality field sphere diameter</td>
<td>0.1 m</td>
</tr>
<tr>
<td>Coil maximum thickness</td>
<td>0.01 m</td>
</tr>
</tbody>
</table>

The spherical solenoid consists of three coil blocks in each half (see Fig. 5). The magnet coils have dipole symmetry. So, $C_{n} = 2, 4, 6 \ldots$ are eliminated by symmetry. There are five coil parameters available to zero five $C_{n}$ coefficients. They are the five block angles $G(2)$ through $G(6)$ shown in Fig. 5. These five block angles can be set so that $C_{n} \approx 0$, for $n = 3, 5, 7, 9$ and 11; and $C_{n} \neq 0$ when $n > 13$. Figure 5 shows the five coil parameters which are varied to eliminate the $C_{3}, C_{5} \ldots$ to $C_{11}$. A sixth parameter $G(1)$, the current density in the coil blocks (which are all assumed to have the same current density), determines the induction of the magnet at its center ($C_{1} = 3.0$ T). Table 2 shows the first guess and final value of each of the six magnet parameters.

Table 2. The first guess and final value for each of the six parameters used to determine the design of a uniform solenoid.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>First Guess</th>
<th>Final Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{1}$ The coil current density ($A m^2$)</td>
<td>$3 \times 10^8$</td>
<td>$3.514 \times 10^8$</td>
</tr>
<tr>
<td>$G_{2}$ 2nd coil angle, Block 1 (deg.)</td>
<td>60.00</td>
<td>58.3490</td>
</tr>
<tr>
<td>$G_{3}$ 1st coil angle, Block 2 (deg.)</td>
<td>50.00</td>
<td>55.1390</td>
</tr>
<tr>
<td>$G_{4}$ 2nd coil angle, Block 2 (deg.)</td>
<td>45.00</td>
<td>39.2733</td>
</tr>
<tr>
<td>$G_{5}$ 1st coil angle, Block 3 (deg.)</td>
<td>30.00</td>
<td>30.3888</td>
</tr>
<tr>
<td>$G_{6}$ 2nd coil angle, Block 3 (deg.)</td>
<td>25.00</td>
<td>21.0544</td>
</tr>
</tbody>
</table>

![Figure 5. A good field quality spherical solenoid.](image-url)
Table 3 shows a value of $C_n$ at the surface of a 0.1-m diameter sphere for each $n$ between 1 and 20. The table shows the value for the first guess case and the final case.

Table 3. The numerical value of the $C_n$ coefficients (units not given) at the surface of a sphere 0.1 m in diameter.

<table>
<thead>
<tr>
<th>$n$</th>
<th>First Guess</th>
<th>Final Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.25926</td>
<td>3.00000</td>
</tr>
<tr>
<td>3</td>
<td>$-7.347 \times 10^{-2}$</td>
<td>$-1.849 \times 10^{-5}$</td>
</tr>
<tr>
<td>5</td>
<td>$-3.107 \times 10^{-4}$</td>
<td>$-1.499 \times 10^{-5}$</td>
</tr>
<tr>
<td>7</td>
<td>2.987 $\times 10^{-4}$</td>
<td>$1.943 \times 10^{-8}$</td>
</tr>
<tr>
<td>9</td>
<td>$-5.671 \times 10^{-5}$</td>
<td>$2.463 \times 10^{-9}$</td>
</tr>
<tr>
<td>11</td>
<td>$-1.929 \times 10^{-6}$</td>
<td>$1.943 \times 10^{-8}$</td>
</tr>
<tr>
<td>13</td>
<td>$-1.456 \times 10^{-6}$</td>
<td>$-1.849 \times 10^{-5}$</td>
</tr>
<tr>
<td>15</td>
<td>$-2.036 \times 10^{-8}$</td>
<td>$-1.499 \times 10^{-7}$</td>
</tr>
<tr>
<td>17</td>
<td>$2.914 \times 10^{-8}$</td>
<td>$1.943 \times 10^{-8}$</td>
</tr>
<tr>
<td>19</td>
<td>$4.611 \times 10^{-10}$</td>
<td>$2.463 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

*Note $C_n = 0$ when $n = 2, 4, 6, 20$.

The solution given in Tables 2 and 3 can be obtained theoretically in a superconducting solenoid. It should be pointed out that real-life superconducting magnets have finite size winding errors. It might be possible to build a real magnet which has a field uniformity of $10^{-5}$ or better. It should be noted that diamagnetic currents will cause large distortions in the field. As a result, field uniformities better than $10^{-5}$ may be difficult to obtain in real life without the use of special correction winding to get rid of the effects of residual field due to diamagnetic currents.

(b) Correction coils for a magnet with flat poles. Correction coils can be designed so that they produce $C_n$ coefficients of a certain magnitude. There is a need for such correction coils in some of the solenoids now being designed for use in colliding beam experiments. These magnets have superconducting coils which are bounded at the ends by flat iron poles. The solenoid is supposed to produce a perfectly uniform field. Misalignment, saturation of the iron poles, and other effects may introduce errors in the field of a part or two in a thousand. Correction coils can remove 90 percent of these errors.

Figure 6 shows a system of eight coils which are to produce field corrections. The coils, depending on how they are powered are supposed to produce $C_1, C_2, C_3, C_4, C_5, C_6$, and $C_7$ while eliminating all other $C_n$ below $n = 9$. The eight coils are symmetrically placed on either side of the midplane, and evenly spaced between the iron poles ($\mu = \infty$) which are infinite planes (see Fig. 6). Table 4 gives the basic parameters of the correction coil blocks.

Table 4. Basic parameters of the correction coils.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correction coil spacing</td>
<td>0.4 m</td>
</tr>
<tr>
<td>Iron pole spacing</td>
<td>3.2 m</td>
</tr>
<tr>
<td>Correction coil radius</td>
<td>1.2 m</td>
</tr>
<tr>
<td>Value of $C_n$ desired at surface of reference sphere</td>
<td>0.005 T</td>
</tr>
<tr>
<td>Radius of reference sphere</td>
<td>1.0 m</td>
</tr>
</tbody>
</table>
Figure 6. A correction coil problem.

Symmetry is used to do the calculation. Dipole symmetry is used for C1, C3, C5 and C7 and quadrupole symmetry is used for C2, C4 and C6. The parameter which one solves for is the current in each of the coils. There is a different current distribution in each of the coils for each correction \( C_n \) produced. The simultaneous equations generated by the computer are linear, so no iteration is required.

Table 5 shows current in each of the eight coils when the desired coefficient is produced. Table 6 shows the values of other coefficients when the desired one is produced.

Table 5. The current in the eight correction coils when a desired multipole is generated on the surface of a 1.0 m radius reference sphere.

<table>
<thead>
<tr>
<th>Coil</th>
<th>Current (A)</th>
<th>( n = 1 )</th>
<th>( n = 2 )</th>
<th>( n = 3 )</th>
<th>( n = 4 )</th>
<th>( n = 5 )</th>
<th>( n = 6 )</th>
<th>( n = 7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coll 1</td>
<td>1552</td>
<td>-6168</td>
<td>12192</td>
<td>-20912</td>
<td>27020</td>
<td>-34516</td>
<td>28208</td>
<td></td>
</tr>
<tr>
<td>Coll 2</td>
<td>1640</td>
<td>-488</td>
<td>-4244</td>
<td>13968</td>
<td>24208</td>
<td>38392</td>
<td>-34832</td>
<td></td>
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<tr>
<td>Coll 3</td>
<td>1572</td>
<td>-1464</td>
<td>-992</td>
<td>1376</td>
<td>312</td>
<td>-9372</td>
<td>11980</td>
<td></td>
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<tr>
<td>Coll 4</td>
<td>1596</td>
<td>-280</td>
<td>-2376</td>
<td>1764</td>
<td>1176</td>
<td>36</td>
<td>-2484</td>
<td></td>
</tr>
<tr>
<td>Coll 5</td>
<td>1596</td>
<td>280</td>
<td>-2376</td>
<td>-1764</td>
<td>1176</td>
<td>-36</td>
<td>-2484</td>
<td></td>
</tr>
<tr>
<td>Coll 6</td>
<td>1572</td>
<td>1464</td>
<td>-992</td>
<td>-1376</td>
<td>312</td>
<td>9372</td>
<td>11980</td>
<td></td>
</tr>
<tr>
<td>Coll 7</td>
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<td>-13968</td>
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<td>-38392</td>
<td>-34832</td>
<td></td>
</tr>
<tr>
<td>Coll 8</td>
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<td>6168</td>
<td>12192</td>
<td>20912</td>
<td>27020</td>
<td>34516</td>
<td>28208</td>
<td></td>
</tr>
</tbody>
</table>

Desired \( C_n \):

\[
\begin{align*}
C_n &= 0.005 \\
C_n &= 0.005 \\
C_n &= 0.005 \\
C_n &= 0.005 \\
C_n &= 0.005 \\
C_n &= 0.005 \\
C_n &= 0.005
\end{align*}
\]

*On the surface of the reference sphere which is 1.0 meter in diameter.

A correction coil scheme similar to the one shown here is expected to be used on the TPC solenoid which is to be built in 1978. This magnet is supposed to produce a magnetic field which is good to a couple of parts in ten thousand. In the real TPC magnet, the iron poles are not infinite planes of infinite permeability. A technique such as this is expected to provide first order correction.
Table 6. The values of various $C_n$ coefficients when the desired correction $C_n$ is turned on.

<table>
<thead>
<tr>
<th>n</th>
<th>$C_{nD} = 1$</th>
<th>$C_{nD} = 2$</th>
<th>$C_{nD} = 3$</th>
<th>$C_{nD} = 4$</th>
<th>$C_{nD} = 5$</th>
<th>$C_{nD} = 6$</th>
<th>$C_{nD} = 7$</th>
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<td>0.00500</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>2</td>
<td>—</td>
<td>0.00500</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>3</td>
<td>—</td>
<td>—</td>
<td>0.00500</td>
<td>—</td>
<td>—</td>
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<td>—</td>
</tr>
<tr>
<td>4</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>0.00500</td>
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<tr>
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<td>—</td>
<td>—</td>
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<td>—</td>
<td>0.00500</td>
<td>—</td>
<td>—</td>
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<tr>
<td>6</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>0.00500</td>
<td>—</td>
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<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
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<td>—</td>
<td>—</td>
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<td>—</td>
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<tr>
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<td>—</td>
<td>—0.00020</td>
<td>—</td>
<td>—0.00128</td>
<td>—</td>
<td>—0.00258</td>
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<tr>
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<td>—</td>
<td>—0.00009</td>
<td>—</td>
<td>—0.00092</td>
<td>—</td>
<td>—0.00252</td>
<td>—</td>
</tr>
<tr>
<td>11</td>
<td>0.00009</td>
<td>—</td>
<td>—0.00020</td>
<td>—</td>
<td>0.00014</td>
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<td>—0.00001</td>
<td>—</td>
<td>0.00064</td>
<td>—</td>
<td>0.00091</td>
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<tr>
<td>13</td>
<td>—0.00017</td>
<td>—</td>
<td>0.00039</td>
<td>—</td>
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<td>—</td>
<td>0.00174</td>
</tr>
</tbody>
</table>

*On the surface of a reference sphere 1.0 m in radius

References

